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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

DIVERSION TUNNEL AND POWER CONDUIT OF NANTAHALA HYDROELECTRIC DEVELOPMENT

By D. J. Bleifuss,[1] Assoc. M. ASCE

Synopsis

The salient features of the diversion tunnel and power conduit of the Nantahala Hydroelectric Development, in North Carolina, constructed in 1940, 1941, and 1942, are described in this paper. The tunnel was used to carry the flow of the river during construction, and in the finished project provides emergency release from the reservoir. A self-removing temporary closure gate was used. Permanent control is attained by narrow and long fixed-wheel gates operating in a rigid arrangement of guides and liner. An energy dissipator prevents damage to the unlined tunnel below the control gates.

The power conduit carries water from the reservoir to the powerhouse. There are lined and unlined sections of pressure tunnel, a pipe line 9 ft in diameter crossing Dick's Creek, a fully welded penstock carrying a static head of slightly more than 1,000 ft, a surge tank with a restricted orifice, a rock catcher, and an intake. The tunnel was located so as to avoid water loss by seepage.

General Description of Plant

The Nantahala Hydroelectric Plant is situated in the Great Smoky Mountains of Macon County, North Carolina, about 18 miles west of Franklin, the county seat, and 50 miles south-southeast of Knoxville, Tenn. It was designed and built in 1940 to 1942, although preliminary designs had been made years before. Approximately 1 mile of tunnel had been completed in 1931. The development takes its name from the small river on which it is located. The valley is narrow and deep, so the Cherokees christened it "Nantahala," meaning the "valley of the noonday sun."

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by May 1, 1950.

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Fig. 1 shows the general layout of the development. A rock and earth-fill dam creates the reservoir. In order to build the dam, it was necessary to drive a diversion tunnel. The power conduit extends from the intake in the reservoir to the powerhouse and has a total length of 29,499.02 ft, or 5.58

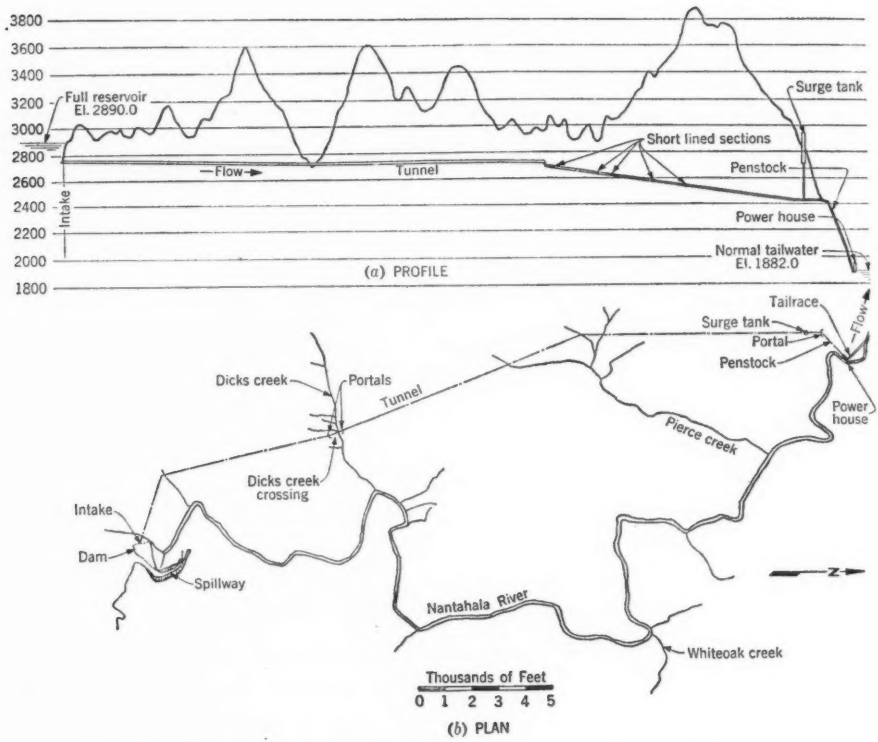


FIG. 1.—GENERAL PLAN AND PROFILE OF NANTAHALA DEVELOPMENT, NORTH CAROLINA

miles, horizontal measurement. Full reservoir water surface is at El. 2890.0, and normal tailwater, at El. 1882.0, so that the maximum static head on the plant is 1,008.0 ft. The powerhouse contains a single Francis-type reaction turbine. The dam[2] and the turbine installation[3] have been described elsewhere.

GEOLOGY

The Great Smoky Mountains form part of the Appalachian system and are relatively old. In the area of the Nantahala development, the rock is nearly all arkose, a variety of sandstone containing much feldspar. Feldspars are silicates of aluminum with potassium, sodium, and calcium present in varying amounts, and the sands were presumably derived from the weathering of feld-

[2] "The Nantahala Earth-Faced Rock-Fill Dam," by J. P. Growdon, *Proceedings*, ASCE (publication pending).

[3] "Nantahala Turbine," by J. P. Growdon, R. V. Terry, and H. H. Gnuse, Jr., *Transactions*, A.S.M.E., Vol. 68, 1946, p. 687.

spathic rocks such as granite. The rock is generally sound, very abrasive, and blocky. When work was begun, no faults were known in the area.

The overburden is a red clay, varying widely in depth, which stands well in its natural condition. Previous experience on another project in similar material had demonstrated no slides need be expected on reservoir slopes.

The Diversion Tunnel

The drainage area above the dam site is about 91 sq miles, and the average annual rainfall, about 70 in. The drainage area is heavily forested, but rugged enough so that runoff is quite rapid. The average annual flow of the Nantahala is 331 cu ft per sec, and the low flow perhaps 40 cu ft per sec; but a flood of 60,000 cu ft per sec may be expected, although none greater than 15,000 cu ft per sec has ever been known to occur. It was decided to provide for the

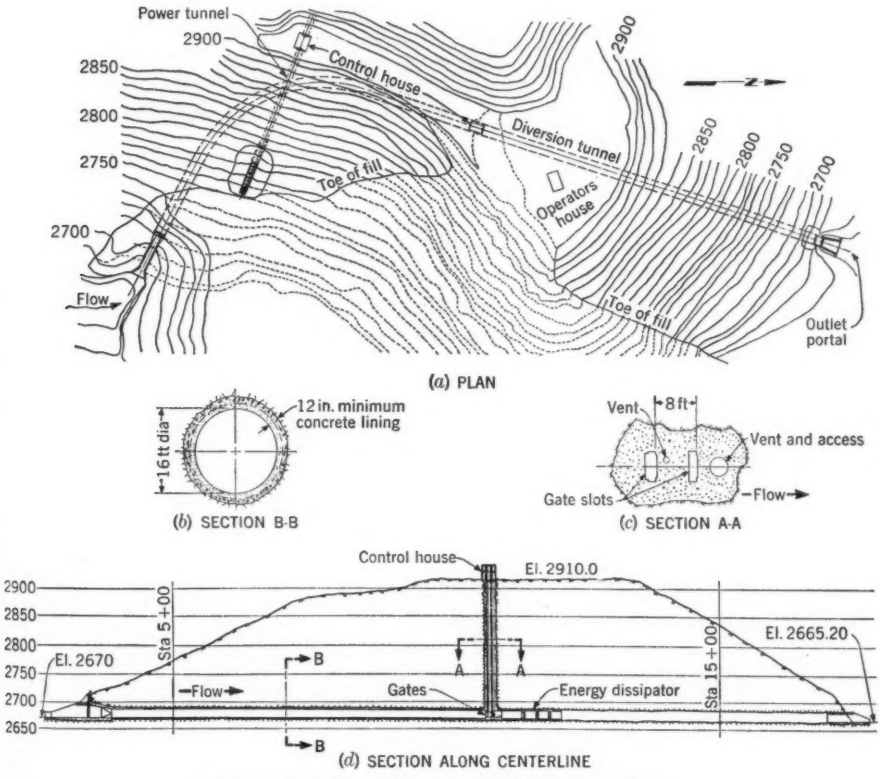


Fig. 2.—General Arrangement of Diversion Tunnel

diversion of a maximum of 7,500 cu ft per sec. Fig. 2 shows the general arrangement of the diversion tunnel.

Four main factors influence the discharge capacity of such a tunnel: (1) The tunnel length; (2) the tunnel area; (3) the head available to force water through the tunnel; and (4) the roughness of the tunnel walls.

The tunnel length is determined by alinement and grade, and these depend on the terrain. A diversion tunnel should be as low as possible to reduce the height of the upstream cofferdam. The entrance must be far enough upstream so that ample room is available for cofferdam construction, and so that currents will not disturb the cofferdam. The exit must be downstream a sufficient distance to prevent the discharge from interfering with the work or from damaging any structure. If possible, the axis of the tunnel, at both entrance and exit, should be at right angles to the surface contours. There should also be enough rock cover so that in driving the abutment of the dam is not weakened.

The remaining three factors are interdependent. The head available depends on the height to which the upstream cofferdam is built, and must equal the sum of velocity head and losses. Relative economics of various combinations of cofferdam height and tunnel diameter must be considered. The Manning formula was used to determine losses; thus,

$$h_f = L \frac{(Q\,n)^2}{1.486\,A\,R^{2/3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which h_f is the friction head loss, in feet; L is the length of conduit, in feet; Q is the quantity of water, in cubic feet per second; n is the Kutter roughness coefficient; A is the cross-sectional area of conduit, in square feet; and R is the hydraulic radius of conduit, in feet.

It was decided that, after construction, the tunnel would be utilized as an emergency outlet, and as a means of providing additional flood discharge capacity. Therefore, it was necessary to design permanent control works. Two alternatives were considered—a control tower in the reservoir at the tunnel entrance, and a shaft about on line with the axis of the dam extending upward from the tunnel to the rock surface. The shaft was chosen (see Fig. 2) as being somewhat more economical and safer. A control tower requires forms and reinforcing steel, and these may be balanced against the cost of sinking the shaft. Although this area is not ordinarily considered as subject to earthquakes, there has been one disastrous earthquake at Charleston, S. C., and a major earthquake in western Tennessee (resulting in the formation of Reelfoot Lake). Natahala is about midway between these places, and a control tower would be more likely to be damaged by an earthquake than a shaft.

It was obvious that since the control would have to be capable of operating under full reservoir head, the control gates and their setting would form a constriction that would reduce the carrying capacity of the tunnel during diversion. Therefore, it was decided that the gates would be installed after diversion, while the reservoir was filling, and that, otherwise, as much work as possible should be completed before diversion was started. Accordingly—the tunnel was driven, the concrete portals were constructed, the tunnel was lined upstream from the control gates, and the control shaft was sunk and lined before diversion.

An interesting arrangement was made for closure of the tunnel after diversion to begin filling the reservoir. A closure gate was required which would

resist the head while the permanent control gates at the base of the shaft were being installed, and which could easily be removed thereafter. The opening at the intake portal was made high and narrow (10 ft by 24 ft) to accommodate a sliding gate. Fig. 3 shows the design of this structure, and Fig. 4 shows the closure gate. The latter had a skin plate on both faces; it was in fact a caisson. It was determined that not more than 150 ft of head could accumulate while the permanent gates were being placed, and the gate was designed accordingly, using a unit stress of 24,000 lb per sq in. It was placed by a portable crane, and water pressure held it firmly in place as long as the tunnel behind it was empty. The concrete portal was provided with a by-pass valve for filling the tunnel after the permanent gates were installed and closed, and with a 3-in. pipe in the roof to allow the trapped air to escape. The theory was that, when the pressures on the two sides of the gate balanced, it would simply float to the surface. The scheme was successful, but the gate, as fabricated, did not have a strictly planar sealing surface, and permanent gate installation was somewhat hampered by leakage into the tunnel.

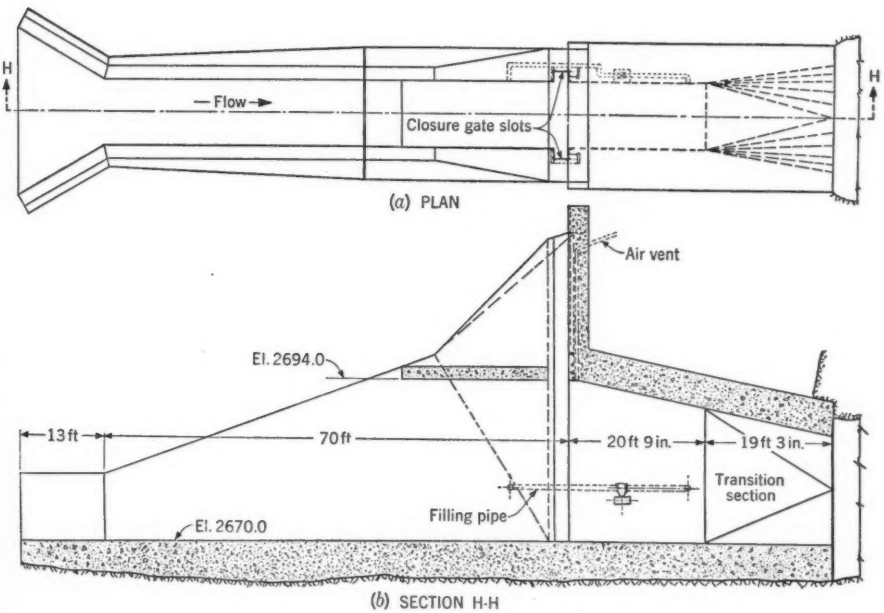


FIG. 3.—INTAKE PORTAL, DIVERSION TUNNEL

Fixed-wheel gates, with cable hoists, were chosen as being the simplest and most economical control. There are two gates, in series; the downstream gate is considered the operating gate, and the upstream one, the emergency gate, although the two are identical. The first installation step was to place the gate guide and liner assembly, shown in Fig. 5. This was a very rigid arrangement, designed in four pieces for easy shipment and handling. However, it

was completely shop assembled and very carefully checked. The field joints were bolted, as it was feared that any welding might disturb the exact alinement of roller paths and seals. The assembly comprised a bell mouth, the pocket for the upstream gate, a tapered section, the pocket for the downstream gate, and another tapered section. Air vents were provided downstream from each gate, since at partial gate openings considerable air is required to prevent irregularities in flow. Seal and roller paths are of stainless steel.

Fig. 6 shows the details of concrete in the shaft. The air vents are extended to the top of the shaft, the downstream vent being made 3.5 ft in diameter so that it may be used as a manhole. The vents terminate in an air passage under the gatehouse floor, and open to the outer air, in order that no partial vacuum can ever be created within the gatehouse. Wood water stops were provided at all horizontal lift joints.

The fixed-wheel control gates, shown in Fig. 7, were made long and narrow. There are two main advantages in this shape—the span between wheels is short, and the gate has less tendency to chatter. The gates were filled solidly with concrete, and will close by their own weight under full head. The rubber seals were of a new design and, when the gates were first closed, these seals were absolutely watertight. However, it has been found that when the gates are operated under full head, the flow over the top of the gate and around the sides above the ends of the seal plates damages the rubber seals. Apparently, the seal plates should have been extended upward far enough so that no flow could pass the side seals until the gate was fully opened. The top seal should have been protected by suitable baffle plates to prevent water under high pressure and velocity from passing the seal.

The gatehouse at the head of the shaft has two stories. The upper story carries the gate hoists, and the lower story is high enough so that a completely hoisted gate will hang clear of the floor and is therefore entirely accessible. Each gate is lifted by two wire ropes $1\frac{5}{8}$ in. in diameter. Hoists are electrically driven, and as no permanent power line to the dam was contemplated, a prime mover was provided in the shape of a 10-kw, gasoline-driven generator set. In this mountainous country, subject to intense thunderstorms, a transmission line is likely to give trouble, and the generator set would have been required as a stand-by source of power in any event.

It was known that only infrequent use would be made of this tunnel for emergency discharge. However, it was feared that a jet of water issuing from the gate under a 213.5-ft head would tear rock out of the walls of the unlined tunnel. Experiments were made, and an energy dissipator was designed to destroy the energy of this jet. Details are shown in Fig. 8. Just downstream from the gate is a steel-armored deflector to split the jet and throw the halves upward and outward. This deflector is in a cylindrical chamber, at the lower end of which is a baffle having a circular hole 9 ft in diameter. Downstream is a second cylindrical chamber, baffled in turn, but the hole is 11.5 ft in diameter. A third chamber follows, the hole this time being 14 ft in diameter. The energy dissipator has had one severe test. It was operated continuously

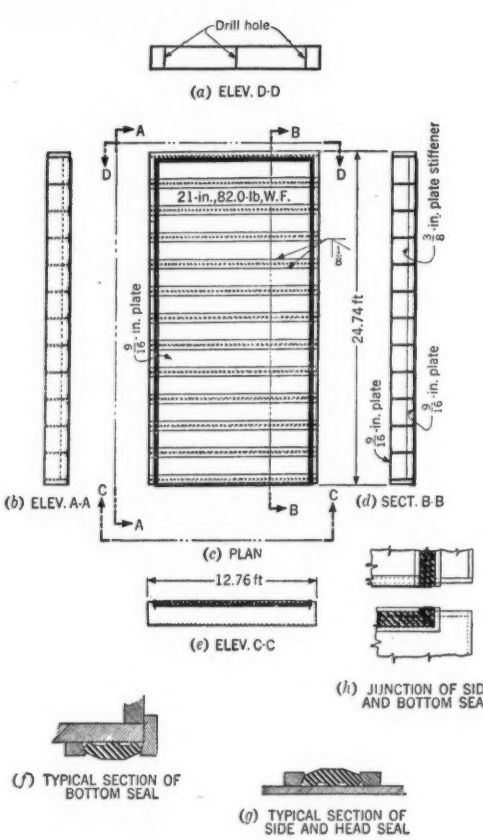


FIG. 4.—CLOSURE GATE, DIVERSION TUNNEL

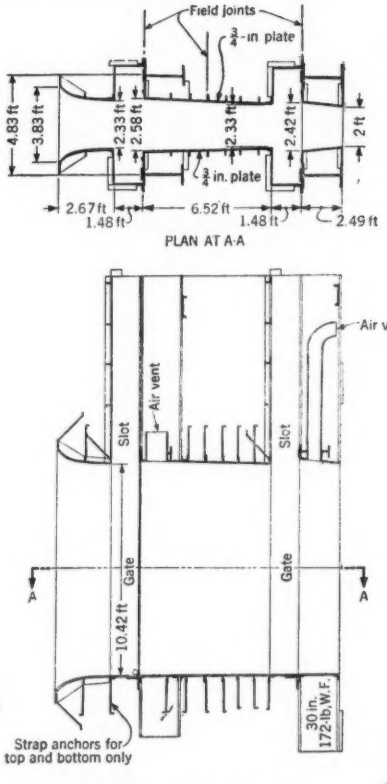


FIG. 5.—CONTROL GATE GUIDES AND LINER

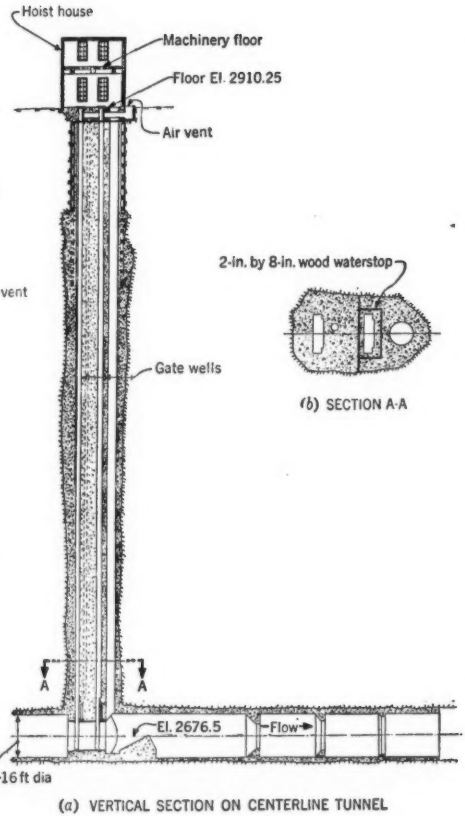


FIG. 6.—CONCRETE, DIVERSION TUNNEL SHAFT

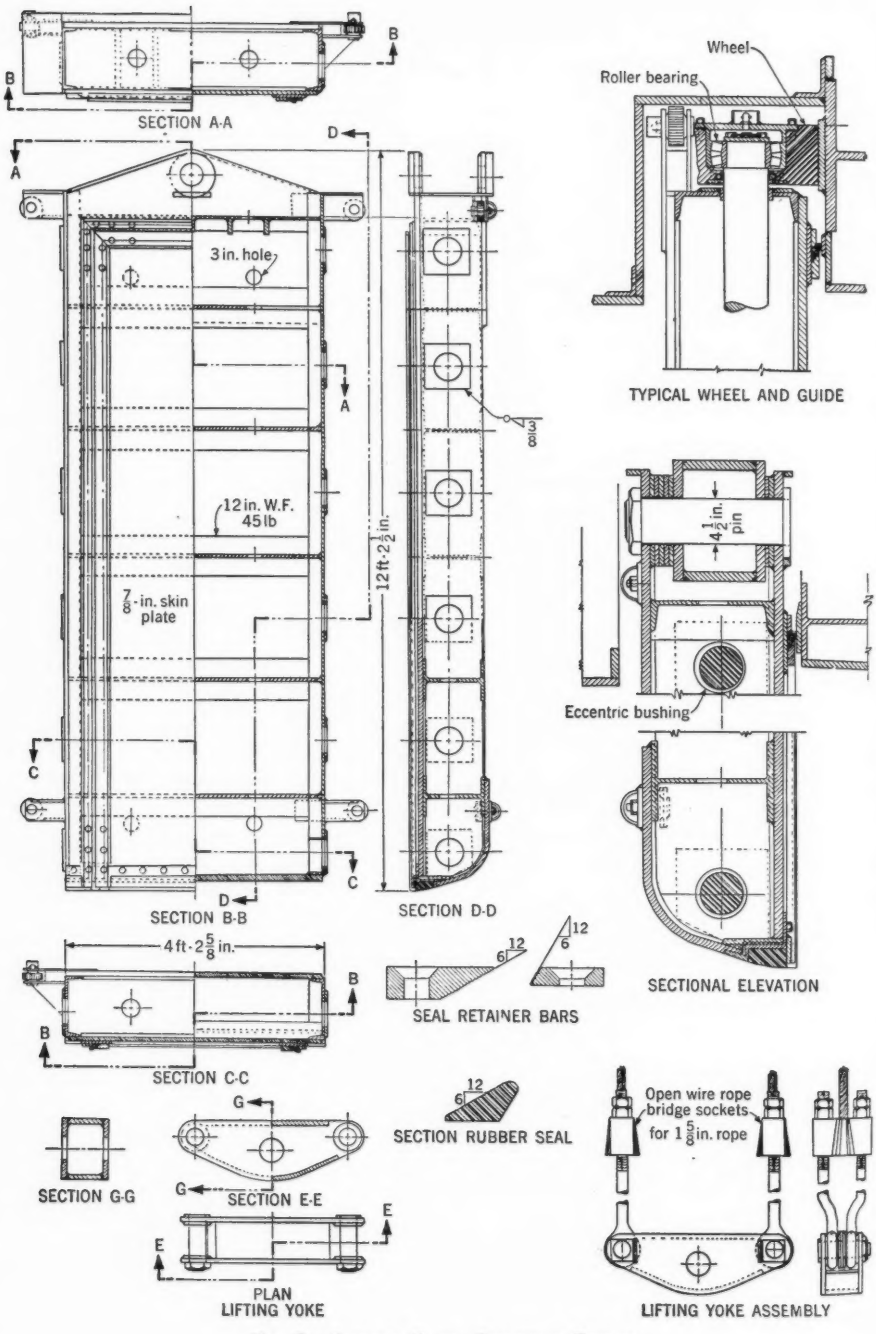


FIG. 7.—CONTROL GATES, DIVERSION TUNNEL

for more than 30 days. Water issued smoothly and quitely, and later inspection showed no evidence of attack on any of the concrete walls or baffles.

The rock was pressure grouted from the shaft, and in a vertical fan-shaped pattern extending downward from the area where the permanent control gates were later set.

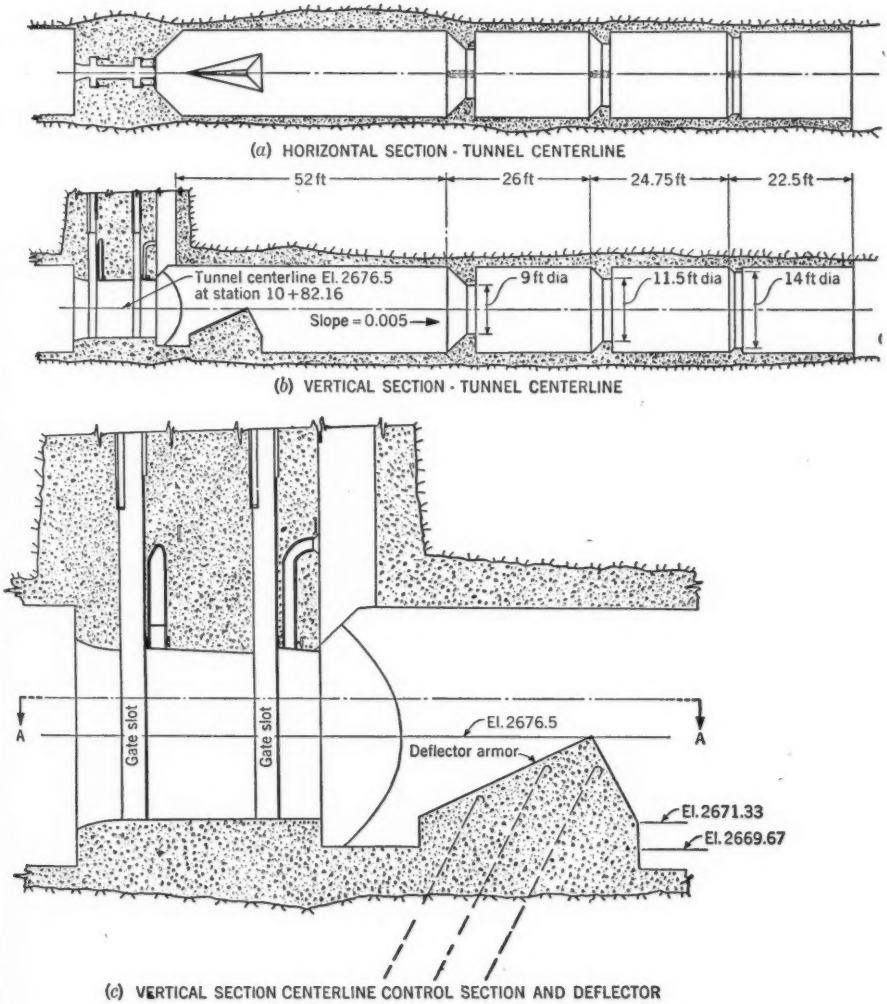


FIG. 8.—ENERGY DISSIPATOR, DIVERSION TUNNEL

THE POWER CONDUIT

General.—The dam was located at the most economical site, considering all factors, such as cost, reservoir capacity, etc. Its foundation is at about El. 2670.0 and full reservoir level is at El. 2890.0. For approximately 10 miles

downstream from the dam site, measured along its course, the river flows in a very deep, narrow, and crooked valley, falling in this distance to El. 1882.0. It then flows out into a wider valley, and its gradient flattens. The mouth of the narrow valley is about 5.5 miles from the dam site, as the crow flies. Preliminary investigation disclosed that the head between the dam site and the valley mouth could be developed by placing the powerhouse at the mouth, and providing a pressure conduit from the dam. The pressure conduit had to be either a steel pipe line or a tunnel, and it was easily demonstrated that it should be mostly tunnel. A pipe line would have been very long, and difficult and expensive to construct and to maintain.

Beginning at the face of the concrete intake, the power conduit comprises:

Item	Description	Length, in feet
1.	A tunnel	8,922.10
2.	A steel pipe line, across the valley of Dick's Creek, a tributary of the Nantahala (length between portals)	573.07
3.	A tunnel	18,706.62
4.	A steel penstock, terminating at the upstream face of the butterfly valve above the turbine	1,247.23
	Total length	29,449.02

(There is a surge tank 1,847.23 ft from the face of the butterfly valve.)

The Tunnel.—When the Nantahala development was started for the first time in 1930 it was decided that the character of the rock would permit the use of unlined pressure tunnels, and in general this proved to be true. It was realized that the rock was not absolutely watertight, and it was obviously more desirable to have water leak into the tunnels than leak out of them. The tunnels were therefore so located that, at any point in unlined sections, the static head on them is less than the vertical distance up to the water table. No loss of water has been observed. The area around the tunnels has been carefully examined for any augmentation of springs or brooks, but nothing of the kind has been found, and it is believed that instead of losing water in the tunnel, actually a little is gained. Alinement and grade have been shown in Fig. 1.

There is a sharp irregularity in the tunnel grade at Station 102 + 92.94. Originally, it was planned to drive from Portal No. 2 on an adverse grade of 0.25% as far as Station 119 + 02.9, and from Portal No. 1 on an ascending grade to intersect at this station. A well hole was sunk at the planned high point to provide a 9-in.-diameter air vent. As the job progressed, it was found advisable to stop driving from Portal No. 1 at Station 102 + 92.94, and complete the tunnel from Portal No. 2. A final grade was adopted as shown. Since water velocities are low, the maximum being only 3.66 ft per sec, the irregularity has a negligible effect on the hydraulic properties of the tunnel, but does make it a little inconvenient to inspect.

Four different cross sections of tunnel were used, as shown in Fig. 9. Work by force account was started at Portal No. 1 in 1930, using a section, 12 ft by 12 ft, with a flat invert and a semicircular roof. It was found that a shot

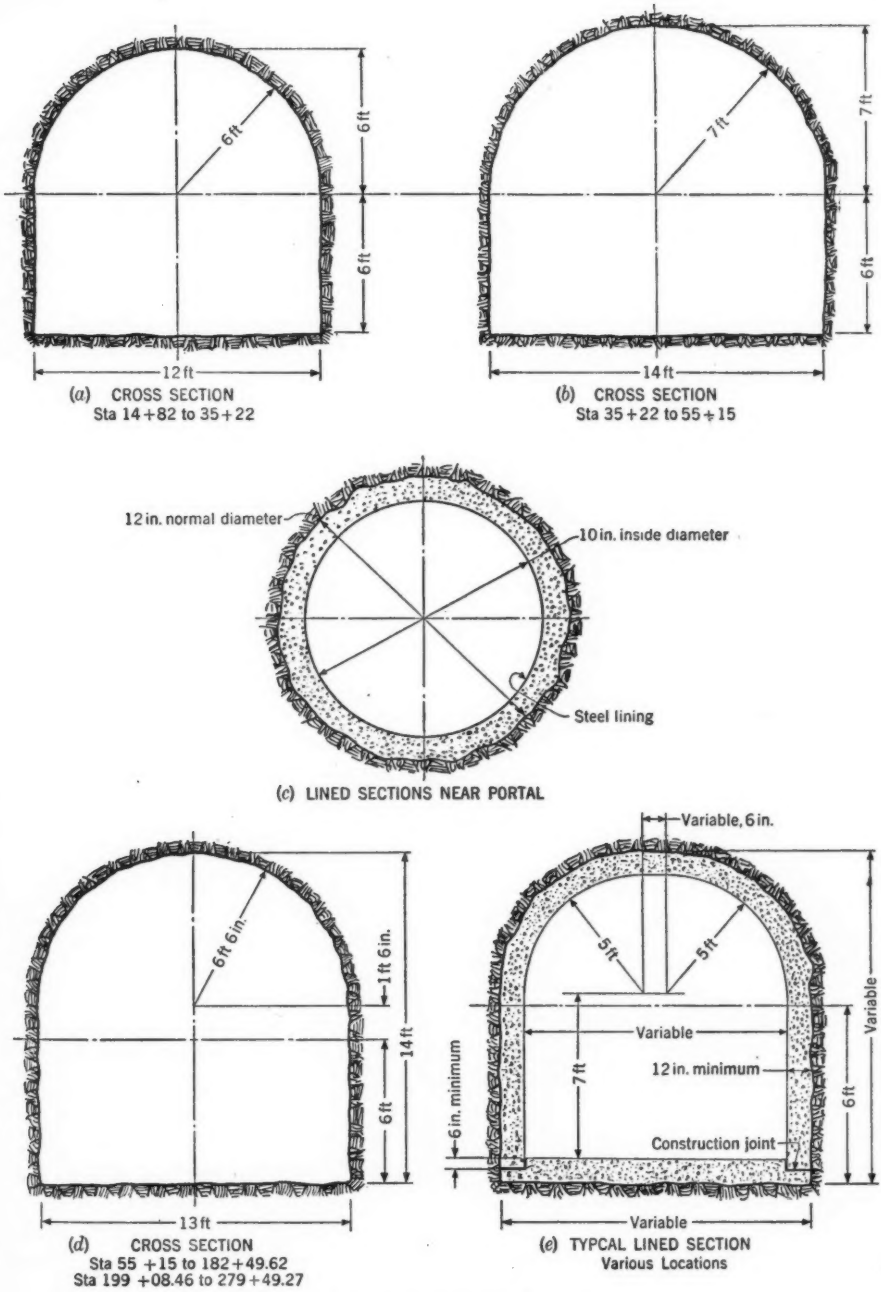


FIG. 9.—TUNNEL CROSS SECTIONS

would only "pull" from 6 ft to 7.5 ft. A section, 14 ft by 13 ft, with a 14-ft flat invert and a semicircular roof was tried, and it was found actually to be cheaper per linear foot than the smaller section. This section was used until work was discontinued because of the economic depression in 1931. When work was resumed in 1940, the contractor elected to drive a section, 13 ft by 14 ft, with a 13-ft flat invert and a semicircular roof. The fourth type of cross section was a 12-ft-diameter circle, used where the tunnels were to be steel lined.

Tunnel headings were driven full face, with no pilot tunnel, except at the one point where difficult ground was encountered. At Station 100 + 42 an apparent fault was encountered (extending to Station 100 + 86) which was filled with an assortment of sand, calcite, talc, and disintegrating arkose. A flow of about 150 gal per min of water developed. It was necessary to use a top heading and bench through this zone, and afterward a heavily reinforced concrete lining was placed for a total length of 215 ft centering on the bad spot.

Although no other bad driving was encountered, it was necessary to place plain concrete linings for short lengths in four other places where narrow zones of poor rock were found.

It has been previously stated that the guiding principle of design was the desirability of having water tend to seep into the conduit, instead of out of it. This tendency is reversed near all portals except at the intake. To prevent water losses, the tunnels were therefore steel lined for varying distances back in from Portals Nos. 1, 2, and 3. At Portals Nos. 2 and 3, the linings were extended back to points where the ground surfaces directly above are at elevations slightly higher than full reservoir level. At Portal No. 1 the ground surface directly over the end of the steel lining is about 400 ft above full reservoir level; but the mountainside to the right of the tunnel is so steep that the minimum distance to the surface is much less. Another reason for extending the lining well back in this portal is that the rock here is generally inferior to that elsewhere on the project.

The Rock Catcher.—The reaction turbine has a speed of 450 rpm, and any solid object passing through with the water would probably cause quite a little damage to the buckets. Also, any sand would damage the turbine seals. It is necessary, therefore, to take all practicable precautions to the end that no solids, large or small, are carried down the power conduit.

When all work in the tunnel had been completed, the walls and crown were inspected carefully and loose material was scaled and, finally, they were hosed down. The invert was then very carefully cleaned. The entire conduit was finally inspected carefully, from end to end, to make certain that nothing loose had been left in it. Then, as the last step, the conduit was flushed out by "cracking" the intake gate, as described subsequently. All these precautions were designed to take care of any solids or debris resulting from construction.

There remained provision to be made for catching any solids originating in the tunnel or in any way entering it during operation. As previously mentioned, normal water velocity is low, and any material moved which is not buoyant must be washed along the invert; but the invert is fairly rough, and the thousands of pockets in it will ordinarily catch and hold such material. How-

ever, whenever the tunnel has been emptied and is refilled, there will be a tendency for any loose material to move along the invert. Fig. 10 shows the rock catcher built to intercept such possible loose material just at the entrance to the steel lining within Portal No. 1. The tunnel was widened on one side, and its invert made 3 ft lower for a length of 100 ft. The invert was smoothed up with concrete, and concrete walls 10 ft apart were built up to the level of the invert of the steel lining. Wood baffles 3 ft apart were placed from wall to wall at right angles to the tunnel axis, forming a series of pockets of dead water, over which any solid material would have to pass to enter the steel-lined section and the penstock. The baffles were made of planks, 3 in. by 12 in., with keeper bars to prevent their floating up out of the retaining slots. In more than 5 years of operation, apparently no solid material has ever reached the turbine, and almost no material has accumulated in the rock catcher.

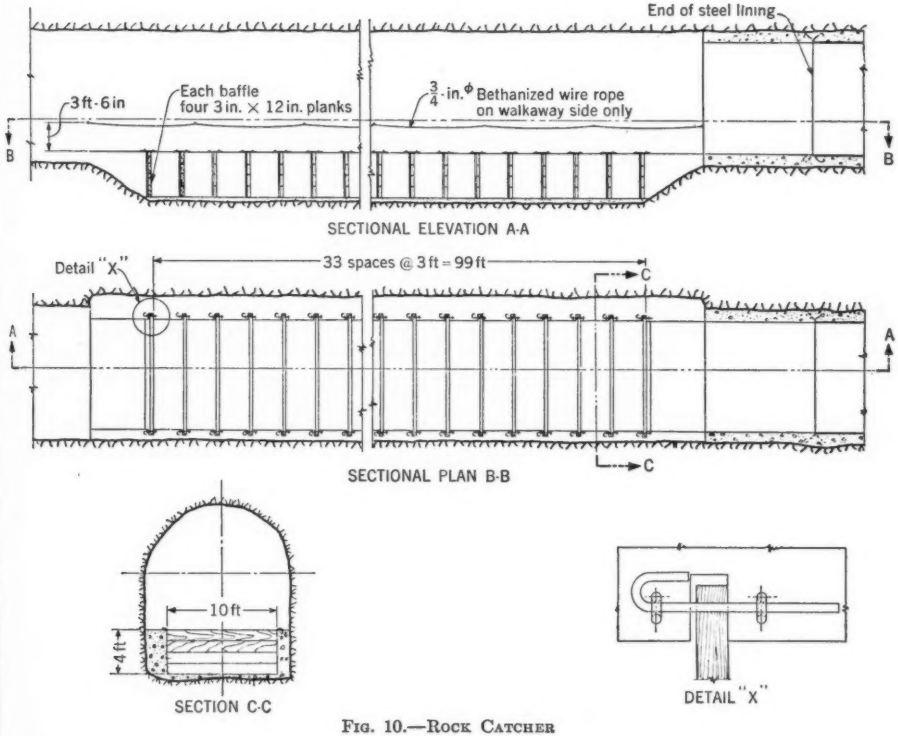


FIG. 10.—ROCK CATCHER

The Intake.—The upstream entrance to the power tunnel is on the side hill, about 70 ft above the bottom of the original river channel. The side hill here is covered with soil and decayed rock. Previous experience at the Santeetlah Reservoir in North Carolina had demonstrated that this material would not slide after submergence, and it was therefore decided that it should be left in place as a matter of economy, and disturbed as little as possible. The control scheme is generally similar to that of the diversion tunnel, and was chosen for

the same reasons. The control shaft is located about 240 ft from the entrance. It became necessary, therefore, to design an intake structure at the entrance. To disturb the overburden as little as possible, this concrete structure was so designed that when completed only a long narrow slot would show on the slope of the mountain. The structure is in two distinct pieces—the one nearest the portal being founded on solid rock and the outer one on the talus slope. Details are shown in Fig. 11. The construction adopted permitted bracing the walls against each other, and heavy retaining walls were obviated. The tunnel is lined for a distance of 40 ft in from the parapet face, to a point where the crown is sound rock.

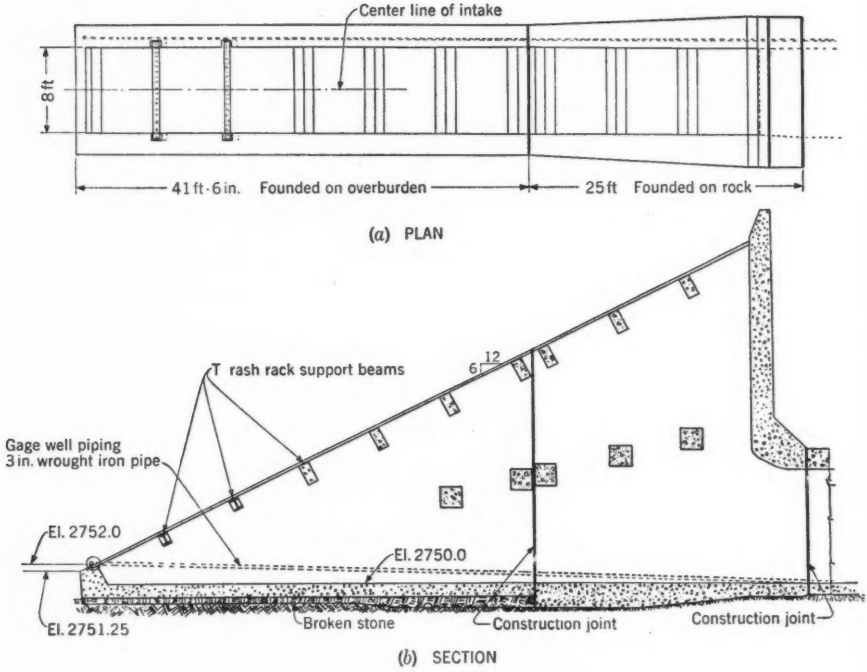


FIG. 11.—POWER TUNNEL INTAKE

The sloping intake opening is covered with a steel trash rack, in sections, laid loosely on steel and concrete beams. The racks were designed for a differential head of 10 ft, and the supporting beams for a differential head of 20 ft. The gross area is about 532 sq ft for a contemplated flow of 600 cu ft per sec, and the net area, about 70% as great. Since the racks can often be reached during low water seasons, no devices for raking or hoisting were provided.

The Intake Control.—General arrangement of the intake is shown in Fig. 12. Two fixed-wheel gates, in series, are placed at the foot of a shaft intersecting the tunnel. The gates themselves are very similar to the control gates installed in the diversion tunnel. The guides were not as rigidly made, however,

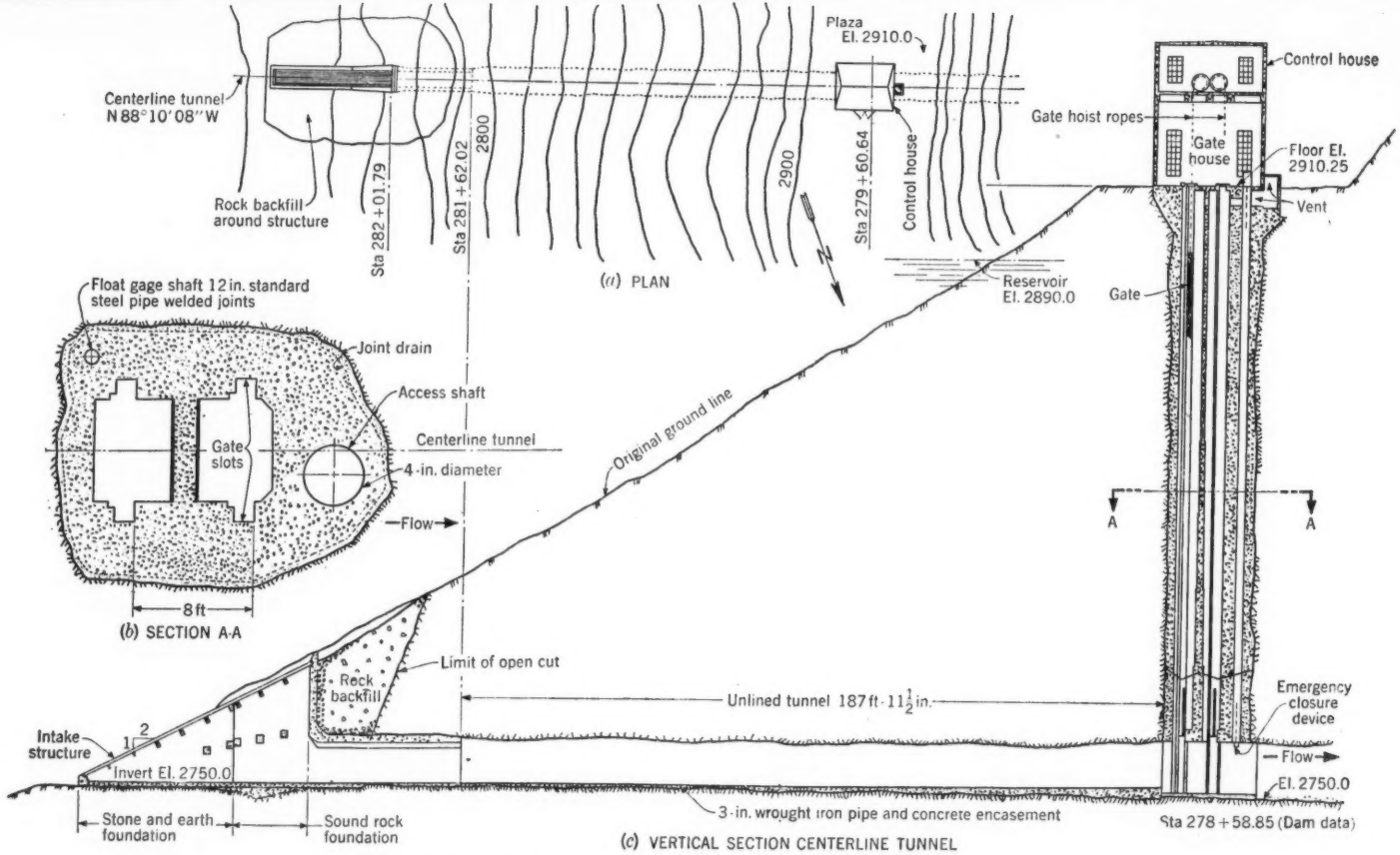


FIG. 12.—GENERAL ARRANGEMENTS, POWER TUNNEL INTAKE

and were slightly warped somehow in the installation, and it was necessary to do a very expensive field grinding job on the seal and roller paths. The rubber seals, similar to those on the diversion control gates, have proved entirely satisfactory here.

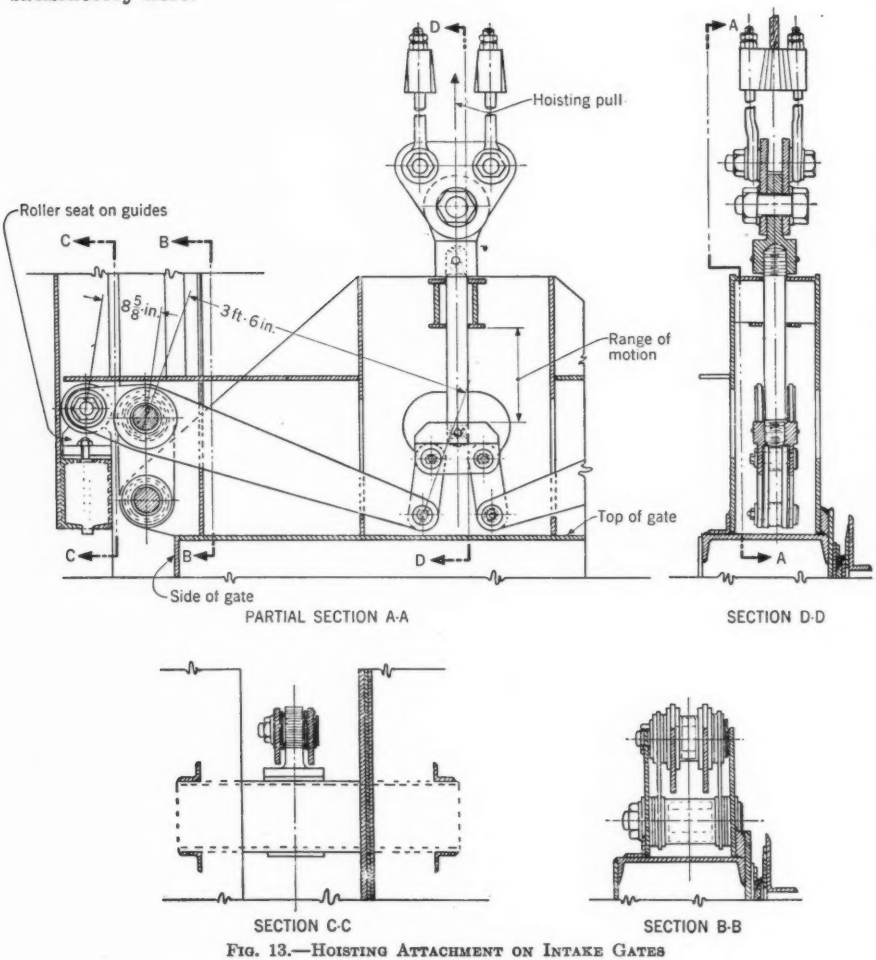


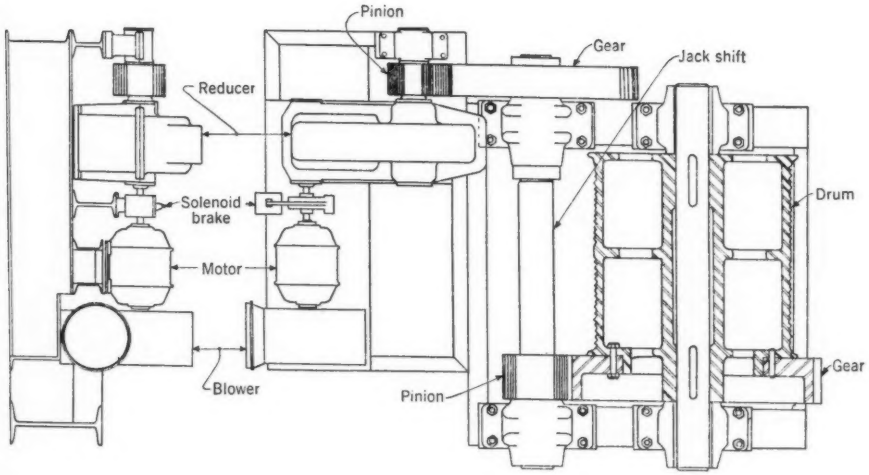
FIG. 13.—HOISTING ATTACHMENT ON INTAKE GATES

The concrete wall between the gate wells was made too thin, and therefore leaked considerably. It has been necessary to repair it.

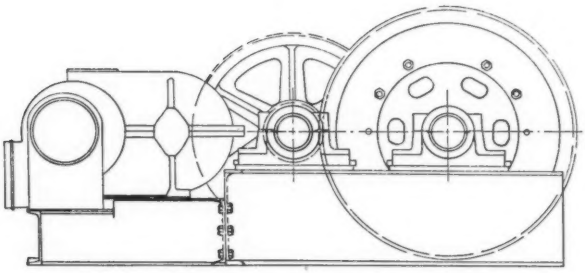
An unusual feature is the method of "cracking" the gates—that is, opening them only a little. When the power conduit has been drained for inspection or repair, it is desirable to refill it slowly. This may be done by a small by-pass gate or by "cracking" the regular gate. The danger in using the latter method is that it is not foolproof; the gate may be opened too wide, and the conduit filled too rapidly. The arrangement adopted guards against this possibility.

Each of the gates is hoisted by two $1\frac{1}{2}$-in. ropes, and will close under its own weight. The hoist provided is strong enough to lift the weight of the

gate, but not strong enough to overcome gate weight plus friction under pressure, and if the hoist ropes were fixedly connected to the gate, it could not be opened. The hoisting force is multiplied so that it is sufficient for "cracking," by connecting the hoist ropes to a pair of levers having a limited range of motion. The levers are hinged on the gate, but their fulcrums are on the gate guides, and the hoisting force is multiplied by the ratio of distances from fulcrum to hoist rope and hinge, respectively (see Fig. 13). This ratio is 5.67. When



(*a*) PLAN · PARTLY SECTIONAL



(*b*) ELEVATION

FIG. 14.—INTAKE GATE HOIST

hoisting tension is put on the wire ropes, the long ends of the levers move through a vertical distance of 14 in. before they hit a rigid stop on the gate, and, in so doing, lift or "crack" the gate 2.47 in. Once the levers hit the stop, the hoist ropes in effect become fixedly attached to the gate, and the hoist is not strong enough to lift farther until the pressures on both sides of the gate are the same.

Another noteworthy feature is a simple and foolproof brake on the hoist for lowering gates. It is automatic in operation, and made of standard stock

parts. It consists of an ordinary air blower direct connected to the motor shaft, in such fashion that the blower runs backward when hoisting. Fig. 14 shows the arrangement, and its operation is explained as follows: Assume that the gate is hoisted, the hoist is held stationary by a solenoid brake, and that no power is available. Release the brake, and the hoist starts to overhaul, the gate descending. As the gate picks up speed, the hoist runs faster, and the blower begins to absorb power. The gate accelerates until it develops power at the same rate as the blower absorbs it, and thereafter the speed of descent is constant. Since the power that a blower requires varies as the cube of its speed, it is easy to design so that the speed of free descent is always less than three times the hoisting speed, which is usually about all the motor will stand. There are only two precautions to be taken—the motor must be checked for the maximum speed, and the blower discharge and intake must be so screened that they cannot be closed accidentally, since a closed blower acts in the same way as a centrifugal pump, and requires no power when it does no work.

All parts of the hoist are made strong enough to stand the stalling torque of the motor, or to withstand the maximum pull on the ropes.

To illustrate the action of the "cracking" device, and the blower, the following data are given:

A. Roller and Roller Bearing Friction.—In the formula—

$$F = \frac{P}{R}(f_r + f_a r) \qquad (2)$$

—the following values applying to rollers were:

Symbol	Description	Value
F	Vertical force, in pounds, found necessary to overcome friction	6,280
P	Horizontal pressure on gate, or load on wheels, in pounds	735,000
R	Radius of wheels, in feet	0.645
f_r	Coefficient of rolling friction for roller bearings	0.003
f_a	Coefficient of axle friction	0.01
r	Radius to center of roller bearings, in feet	0.25

B. Lift Necessary to "Crack" Gate.—The total weight required to "crack" the gates is as follows:

Item	Description	Pounds
1	Weight of gate, filled with concrete, in air	38,700
2	Weight of wire rope	650
3	Downward hydraulic pressure, on top of gate	85,000
4	Roller and roller bearing friction	6,280
5	Seal friction, caused by mechanical deflection of seals, assuming a friction coefficient of 1.0	9,130
6	Seal friction, caused by water pressure on seals	10,900
	Total	150,660

Dividing 150,660 lb by the lever arm ratio of 5.67, the rope pull is 26,600 lb. Thus, the hoist exerts somewhat less force in "cracking" than in lifting the dead weight of the gate.

C. Tension in Hoist Ropes, Gate Closing Under Pressure, Before "Cracking" Device Functions.—The components of force to be considered are:

Item	Description	Pounds
1	Downward forces exerted on gate—	
	a. Weight of gate	38,700
	b. Downward hydraulic pressure	83,000
	Total	121,700
2	Forces resisting downward motion—	
	a. Roller and roller bearing friction	6,280
	b. Seal friction, caused by mechanical deflection	9,130
	c. Seal friction, caused by water pressure	10,900
	Total	26,310
3	Tension in hoist ropes is the difference between items 1 and 2	95,390

D. Tension in Hoist Ropes, Gate Closing Under Pressure, Almost Closed, and "Cracking" Device Functioning.—The component forces for this case are:

Item	Description	Pounds
1	Downward forces exerted on gate—	
	a. Weight of gate	38,700
	b. Pressure on overhang of head seal	5,370
	Total	44,070
2	Forces resisting downward motion (as in item C-2)	26,310
3	Tension in hoist ropes is the difference between items 1 and 2, or 17,760 lb; divided by 5.67 the rope pull is	3,100

The size of the hoist ropes was determined by item C-3. Two $1\frac{1}{8}$-in. wire ropes were used, with a total breaking strength of 205,000 lb. Thus, the factor of safety under emergency closing conditions is $\frac{205,000}{95,390}$ or 2.15.

The hoist motor must be able to lift the gate in the air, and drive the blower backward at the same time. Given: Motor speed, 870 rpm; a reducer ratio of 230 to 1; gear and pinion ratio, 65 to 15; second gear and pinion ratio, 54 to 13; and total reduction, thus, 4,140 to 1.

The drum speed is then 0.21 rpm, and with a pitch diameter of 42 in., the hoisting speed is 2.31 ft per min. The motor must put out, merely to hoist the gate in the air, 3.28 hp, computed as follows—$P = \frac{2.31 \times 39.350}{33,000 \times (0.95)^2 \times 0.93}$ = 3.28. This value takes account of the fact that the reducer has an efficiency of 93%, and each gear and pinion combination, 95%.

The blower requires 2.0 hp at 825 rpm running forward, and one eighth as much running backward. At 870 rpm it requires 0.29 hp running backward, computed as follows: $P = \frac{2}{8} \times \frac{(870)^3}{825} = 0.29$. Thus, the motor must put out a total of 3.57 hp.

The speed at which the gate will descend when the solenoid brake is released is computed as follows:

Step	Description
1	Let N equal revolutions per minute of motor, and therefore of blower.
2	Speed of descent can then be expressed, in feet per minute, as $\frac{3.5\,\pi\,N}{4{,}140}$ or $\frac{N}{377}$.
3	Power generated by the falling gate is speed of descent times tension in hoist ropes, divided by 33,000; thus, $P = \frac{N}{377} \times \frac{95{,}390}{33{,}000} = \frac{N}{130.5}$ ..... (3)
4	The power computed by step 3 is reduced by the gear and reducer efficiencies to $P = \frac{N}{155.5}$ ..... (4)
5	The power required to run the blower is $P = \left(\frac{N}{825}\right)^3 \times 2$ ..... (5)
6	Equating Eq. 4 to Eq. 5, and solving for N, $N/155.5 = 2\,N^3/\,(825)^3$; and $N = 1{,}344$ rpm. This speed is conservative, since some friction in the hoist has been neglected.

The control house at the top of the shaft is similar to that at the top of the diversion control shaft. Power for the hoist is furnished by a small gasoline engine-driven generating set. Since the two control houses are only a short distance apart a power line was run between them, so that the generating set in either house may be used to operate either set of gates. Thus, each generating set is a stand-by for the other set.

Pipe Line Crossing Dick's Creek

To preserve reasonable tunnel alinement and grade, it was necessary to come out in the open with the power conduit and carry a steel pipe line across a ravine at Dick's Creek, and Fig. 15 shows the arrangement adopted. This provided the further advantage of additional portals for tunnel driving. Surface rock in this area was poor, and overlaid by a considerable depth of soft ground. Sufficient floodway for Dick's Creek had to be provided between pipe supports. It was decided to make the pipe 9 ft in diameter in the open,

2800
2750
Sta 187+06.62 Pi
Flow
Dicks creek
2700
2750
2750
Sta Pi 192+79.69
2800
2850
S 12° 36' 26" E
(a) PLAN
¾ in. standard pipe tap in reinforcing plate
½ in. hole in shell plate for grouting
2 in. × ¾ in. × 2 in. reinforcing plate
Weld
Shell plate
After grouting insert ½ in. ϕ punching from ⅛ in. plate and fill ¾ ft. hole in reinforcing plate with weld
Holes at 10 ft centers along pipe centerline for holes 12 in. off of bottom centerline of pipe
(b) DETAIL OF GROUTING CONNECTION AT INVERT OF TUNNEL LINING PIPE
Public hwy
Steel lining 437 ft ± 10 ft diameter
Grade – 0.0025
El. 2721.53
Sta 187+06.62 Portal
Reducer
80 ft 9 ft diameter
Expansion joint
Approx soft rock line
Approx hard rock line
Sta 191+12.62 end of concrete cover
18 ft ϕ manhole
Removable section
Sta 192+79.69 Portal
Steel lining 619 ft ± 10 ft diameter
Grade – 0.0040
El. 2721.53
(c) PROFILE

FIG. 15.—DICK'S CREEK PIPE LINE IN NORTH CAROLINA

and it was designed as a continuous beam over six supports, the longest span being 80 ft. Four of the six supports were founded on concrete caps over steel H-beam piling.

Immediately outside each portal was a concentric reducer between 10-ft-diameter tunnel linings and the 9-ft-diameter pipe in the open. A drain valve was provided in the pipe span over the creek, but because of the difference in diameter the tunnel sections do not drain completely, and this is objectionable. It probably would have been better to make all this pipe 10 ft in diameter.

In the last stages of constructing steel pipe lines and penstocks, access to their interiors becomes a matter of first importance. They must be thoroughly cleaned and painted. After a plant has been put in operation, such access is still very important. Repairs are necessary such as repainting, or cleaning out a rockfall in a tunnel. In either case, it is both inconvenient and expensive to be forced to pass men and materials through manholes, and easier access is highly desirable, although manholes should be provided for the purpose of inspection. At Dick's Creek a 14-ft length of pipe was made removable. The reducer at the stream portal forms part of the removable section. At each end the section is flanged (see Fig. 16) and the flanges drilled for $2\frac{1}{2}$-in.-diameter stud bolts about 8 in. apart on the pitch circle. A corresponding flange is placed on the end of the 10-ft-diameter pipe upstream. All flanges are grooved for round rubber packing. It is the detail at the downstream end of this removable section which makes it easily workable. The section of flanged pipe adjacent is only 25 in. long, and terminates in an expansion joint. When the expansion joint is loosened, and the flange bolts are removed, the short section of pipe can be slid away from the removable section, so that the latter has ample room to come out. It is only necessary to put blocking under it, take a few turns of cable around it, and roll it to one side. It was easy to make this provision; unquestionably, it paid for itself during construction, and it will probably pay for itself several times over in the course of one repair job.

The Penstock

The penstock is considered to extend from a point in the tunnel where steel lining begins to the upstream flange of the butterfly valve in the powerhouse, which lies at the foot of a spur jutting out from Rattlesnake Mountain. That part of the penstock in the open runs down along the narrow spine of this spur. Pertinent data on steel pipe is given in Table 1, distances given being along the penstock and not horizontal, and Fig. 17 shows the general arrangement.

Table 1 shows the location of three removable sections. The upper one is generally similar to the one at Dick's Creek previously described, and is provided for the primary purpose of access. The middle one is much the same, except that it is not a reducer, and was provided essentially for a different reason. Although it may afford access, it was essential to the scheme adopted (described subsequently) for flushing out the tunnel invert. The third and lowest removable section is adjacent to the butterfly valve in the powerhouse, and is quite different from the other two (see Fig. 16).

It was not practicable to put in an expansion joint here, but it was necessary to provide some room for possible removal of the butterfly valve. The last

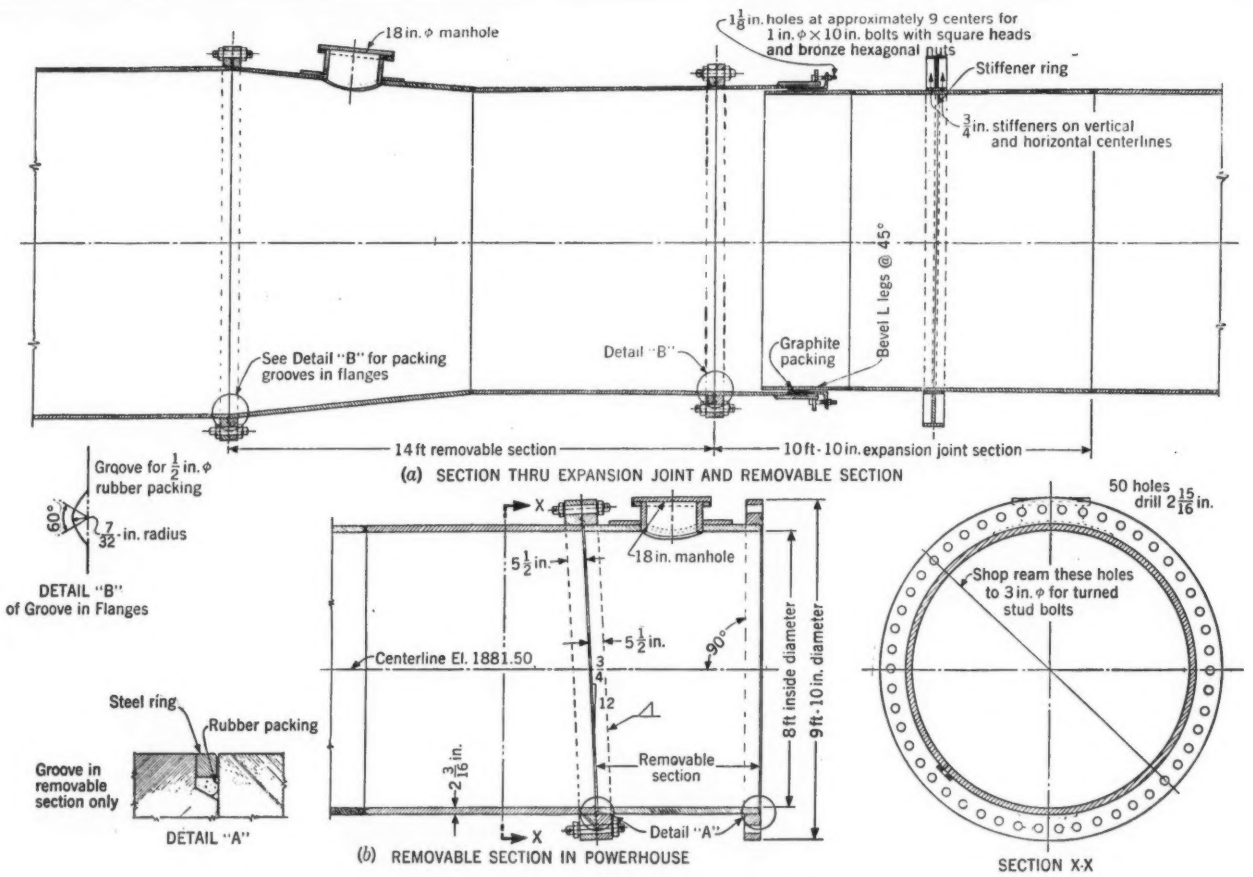

18 in. φ manhole
1⅛ in. holes at approximately 9 centers for 1 in. φ × 10 in. bolts with square heads and bronze hexagonal nuts
Stiffener ring
¾ in. stiffeners on vertical and horizontal centerlines
Bevel L legs @ 45°
See Detail "B" for packing grooves in flanges
Detail "B"
Graphite packing
14 ft removable section
10 ft-10 in. expansion joint section
(a) SECTION THRU EXPANSION JOINT AND REMOVABLE SECTION
Groove for ½ in. φ rubber packing
60°
7/32 in. radius
DETAIL "B" of Groove in Flanges
X
5½ in.
18 in. manhole
5½ in.
90°
Centerline El. 1881.50
3/4
12
8 ft inside diameter
9 ft-10 in. diameter
Steel ring
Rubber packing
Groove in removable section only
2 3/16 in.
Removable section
DETAIL "A"
Detail "A"
X
(b) REMOVABLE SECTION IN POWERHOUSE
50 holes drill 2 15/16 in.
Shop ream these holes to 3 in. φ for turned stud bolts
SECTION X-X

FIG. 16.—EXPANSION JOINTS AND REMOVABLE PIPE LINE SECTIONS

FIG. 17.—PROFILE OF PENSTOCK AND SURGE TANK

section of rigidly held penstock was provided with a flange, 5 ft from the valve flange, inclined at a bevel of $\frac{3}{4}$ to 12 to a line at right angles to the axis of the penstock. The removable section is flanged to match this inclined flange on one end and the valve flange on the other, and is thus slightly wedge shaped, as shown in Fig. 16, the wide part of the wedge being on top. It is under the powerhouse crane, and can be lifted straight out. It is carefully machined to a close fit; it will be shrunk by using dry ice when removal is necessary, and was originally placed using dry ice. The packing at the joints is placed from the interior of the pipe after the removable section is in place. The flanges are recessed as shown; the packing is placed in the recesses and held with a retaining ring sprung into place and tack welded. This type of joint has been very satisfactory.

TABLE 1.—Dimensions of Penstock (See Fig. 17)

Wall thickness (inches) (1)	Length (feet) (2)	Center-line elevation[a] (feet) (3)	Maximum static head[a] (feet) (4)
(*a*) Inside Diameter, 10 Ft			
0.41	204	2,439.35	450.65
0.55	210	....	....
1/2	128	2,427.50	462.50
9/16	24	....	....
5/8	24	....	....
11/16	24	....	....
3/4	384	....	....
13/16	42	....	....
7/8	18	....	....
15/16	18	....	....
1	18	....	....
1 1/16	16	....	....
1 1/8	16	....	....
1 3/16	14	....	....
1 1/4	14	....	....
1 5/16	260	....	....
1 5/16	12.91	2,425.00[b]	465.00
1 5/16	14.00[c,d]	....	....
(*b*) Inside Diameter, 8.5 Ft			
1 3/16	250.83	....	....
1 1/4	22.58	....	....
1 1/4	60.58	2,411.00[e]	479.00
(*b*) Inside Diameter, 8.5 Ft (*Cont'd*)			
1 5/16	64	2,383.05	506.95
1 3/8	64	2,353.52	536.48
1 7/16	70	2,324.00	566.00
1 1/2	63	2,291.70	598.30
1 9/16	56	2,262.64	627.36
1 9/16	7[d]	2,236.80	653.20
(*c*) Inside Diameter, 8 Ft			
1 9/16	101.83	2,233.57	656.43
1 5/8	70	2,186.59	703.41
1 11/16	70	2,154.30	735.70
1 3/4	65	2,122.00	768.00
1 13/16	63.76	2,092.02	797.98
1 7/8	6	2,062.60[e]	827.40
1 7/8	54[c]	2,059.09	830.91
1 15/16	54	2,027.49	862.51
2	57.5	1,995.90	894.10
2 1/16	55	1,962.26	927.64
2 1/8	60	1,930.07	959.93
2 3/16	23.02	1,894.07	995.93
2 3/16	21.22	1,881.50[f]	1008.50
2 3/16	5.00[e]	1,881.50	1008.50

[a] At upper end of run. [b] Anchor at Portal No. 1; vertical and horizontal point of intersection (P.I.). [c] Removable section. [d] Reducer. [e] Anchor at vertical P.I. [f] Vertical P.I. inside powerhouse.

Fig. 17.—Profile of Penstock and Surge Tank

The middle removable section in the penstock was left out when the penstock was built. After all work in the surge tank and power conduit above this point (including painting) was completed, and—so far as was known—the interior had been cleaned scrupulously, the tunnel invert was flushed out after "cracking" the intake gate. The tunnel then filled as far as the apex, and since the water velocity upstream from the apex was low, the flushing did no good in this part of the tunnel so far as moving anything heavier than water

was concerned. It was expected, however, that this procedure would float out any loose pieces of wood overlooked in cleaning; but, from the apex downstream, the water flowed as a turbulent stream down the invert.

Where the removable section had been left out, a temporary funnel section was attached to the penstock, and the discharge was piped over the side of the mountain where it could do no damage. Water was allowed to run in this way for 2 days, and the effluent was carefully observed. At first, this effluent was quite murky because of the rock dust it carried. Any solid object could be heard striking the funnel. From these sounds, it was obvious that the solid particles were all small, with one exception. In spite of the rigorous inspection, one piece of wood, 2 in. by 4 in., and several feet long, came down. When the water became clear, flushing was discontinued, the funnel removed, the removable section inserted, and the power conduit was ready for service.

The penstock was anchored carefully at four points—at the portal, at each of the two vertical curves, and at the lower end. The anchors were designed to resist hydraulic thrust, and to carry the dead weight of pipe. On a steep slope, the latter becomes an appreciable item, since all longitudinal forces are carried by the anchors. For example, the anchor at Station 3 + 00 P (see Fig. 15) resists any longitudinal force because of the weight of pipe between the expansion joints near the portal and about 370 ft below the anchor. At the lower end, just above the powerhouse, the pipe was incased in concrete. The incasement provided a continuous anchor about 238 ft long. Its purpose was to insure that no movement of the penstock could possibly take place which might conceivably disturb the alinement of the generating unit.

Ordinarily the penstock will be drained through the turbine, but it was deemed necessary to provide for draining in case the turbine cannot be run. The penstock is tapped just inside the powerhouse, and a 12-in. pipe, controlled by a gate valve, is taken off and connected to a manifold mounted on a corner of the powerhouse. The manifold feeds four nozzles of 2¾-in. diameter, the nozzles are inclined at 45° with the horizontal, and point up the Nantahala Valley to the right of, and upstream from, the powerhouse. Each nozzle is controlled by a gate valve. When operated, the jets from these nozzles tear themselves to pieces in the air, and the discharged water falls as a heavy mist over an area perhaps 200 ft wide and 600 ft long. It is determined that more than 6,000 hp is dissipated in this way and, since it is all done by air resistance, it sets up a considerable windstorm in the valley.

The Surge Tank

The surge tank was placed as close as possible to the powerhouse, consistent with not having too tall a tank projecting above ground surface. Because of the surface contours, it was not advisable to locate it directly above the tunnel, and it was actually slightly more than 30 ft to the left of the tunnel axis. The riser was inclined with an elbow at the bottom, as shown in Fig. 17. This arrangement permitted the use of a rock bin at the bottom of the riser, so that footings for the riser and surge tank could be excavated without interfering with work in the tunnel.

The surge tank is 600 ft from Portal No. 1, and the most important elevations are as follows:

Item	Location	Elevation (feet)
1	Center line of tunnel at this point	2,426.50
2	Constricted orifice in riser	2,461.80
3	Top of riser, and bottom of surge tank	2,715.00
4	Rock surface, after excavating a flat area	2,840.00
5	Top of surge tank	2,945.00
6	Full reservoir water surface	2,890.00

The riser is steel lined, with an inside diameter of 8 ft. The surge tank has an inside diameter of 19.75 ft, and has a total height of 230 ft, the lower 125 ft of its height being in a shaft excavated in the mountain. Because of the distance between the surge tank and the powerhouse, speed and load regulation cannot be delicate.

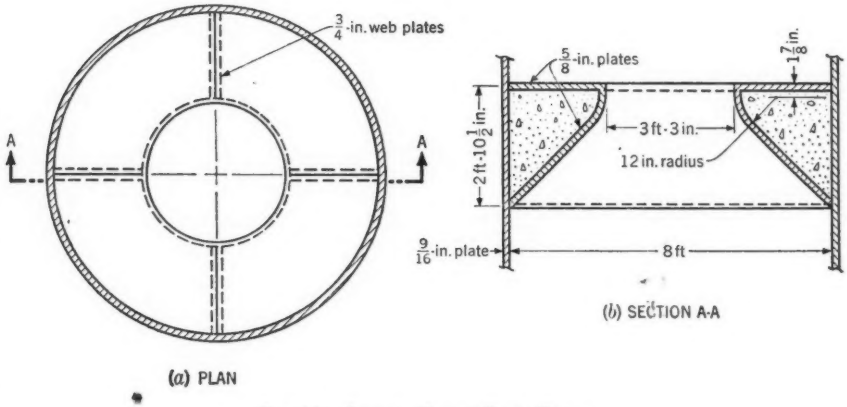


FIG. 18.—PORT IN SURGE TANK RISER

Considering all conditions, either a differential surge tank or one with a restricted orifice was indicated. The differential tank would have required a bottom at El. 2630.0 and, of course, an internal riser. The other type of tank required the development of a restricted orifice with a coefficient of discharge for water passing upward about 50% greater than for water passing downward. Experiments were made on various types of orifices and one finally was designed with discharge coefficients of 0.85 and 0.60. Fig. 18 shows this orifice or port. Computations indicated that, with this orifice, a tank could be built with its bottom at a much higher elevation. The savings in excavation and steel made the orifice type tank much cheaper, and therefore it was used.

DESIGN AND PLACING OF WELDED STEEL CONDUIT AND SURGE TANK

The steel for all tunnel linings, the Dick's Creek pipe line, the surge tank and its riser, and the penstock, conformed to the Specification for Flange Steel,

Serial No. A-70-39, American Society for Testing Materials. This steel was chosen because of its suitability for all welded construction. All welding conformed to the Code for Unfired Pressure Vessels (1938 edition) of the American Petroleum Institute and the American Society of Mechanical Engineers.

This code permits a working stress of 25% of the minimum tensile stress of steel, multiplied by four factors, as follows:

Item	Material	Factor
1	For flange steel	0.97
2	For double-welded butt joints	0.80
3	For stress relieving	1.06
4	For radiographing	1.12

The permissible working stress is thus the product of these four factors and one fourth of 55,000, and proves to be 12,667 lb per sq in. Actually, 12,600 lb per sq in. was used, designing for instantaneous stress resulting from the most severe combination of static pressure, water hammer, and surge. To pipe wall thicknesses thus determined, $\frac{1}{16}$ in. was added as a corrosion allowance.

Two purposes were kept in mind in designing the steel tunnel linings. These were the resistance to stress and the prevention of leakage. Obviously, at the point a pipe enters a tunnel it derives no added strength from any surrounding rock. Also, at some point within the tunnel, good rock becomes capable of taking all the stress. Theoretically, then, the pipe at the entrance must be capable, by itself, of resisting all bursting pressure, and may progressively be made thinner until zero thickness is required. Practical considerations preclude theoretical design. No one knows exactly how to compute the resistance that rock offers to bursting pressure, and pipe must have a certain minimum thickness to make it stiff enough to handle.

Conservative design assumptions were therefore adopted. It was assumed that all bursting pressure would be safely taken by rock at any point in the tunnel where upward pressure was balanced by the sheer weight of rock included between two diverging lines. These lines originated at the ends of the horizontal pipe diameter, extending upward and outward at an angle of 10° from the vertical. Between such points and the portals, linings were designed capable by themselves of resisting all pressures, and to thicknesses so determined the corrosion allowance of $\frac{1}{16}$ in. was added. Beyond these points, pipe wall thickness was decreased by $\frac{1}{16}$ in. per course to a minimum of $\frac{1}{2}$ in. Lengths of such courses varied from 14 ft to 24 ft. The minimum thickness of lining was originally intended to be carried to points at which the vertical distance to the water table would exceed the static head, with the idea that, in parts of the tunnel thus left unlined, water would tend to seep in rather than out, as previously stated. However, in all three places where steel linings were used, they were extended farther than originally intended. At Portal No. 1, the lining was so extended 414 ft; the decision to extend was made during construction and $\frac{1}{2}$-in. plate was not readily available. Plates of 0.41-in. thickness and 0.55-in. thickness were used.

For pipe in the open air, the design of spans and supports was guided by the results of tests reported by R. L. Templin and R. G. Sturm, Members

ASCE, in 1933.[4] These tests indicated that the greatest stresses in a pipe span occurred at supports when it was just full of water under no additional head, and that, if a pipe is stiffened adequately at supports for this condition,

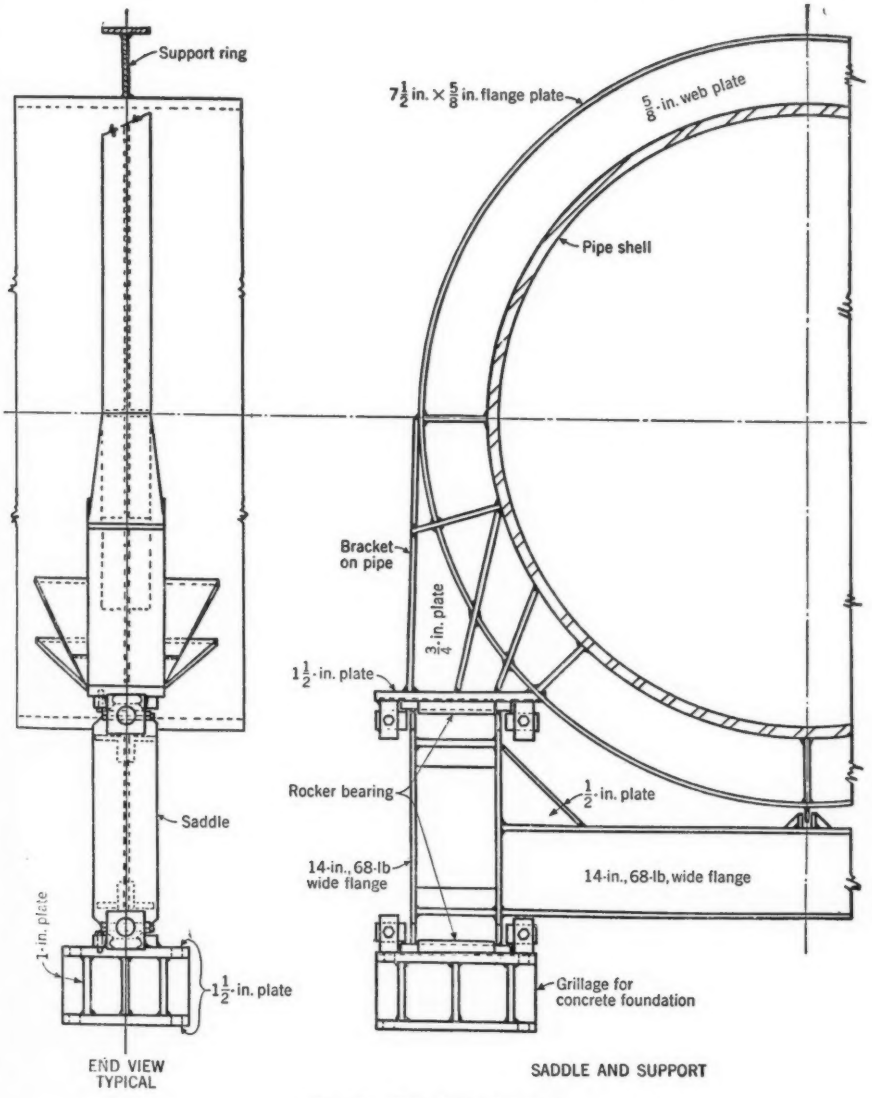


FIG. 19.—PIPE LINE SUPPORTS

it may otherwise be considered as an ordinary cylindrical beam. The tests also indicated a special arrangement of supports that would cause minimum stresses. Fig. 19 shows typical details. The pipe has stiffeners only at sup-

[4] *Transactions*, ASCE, Vol. 98, 1933, p. 154.

ports. Pipe was assumed to act as a continuous beam with no break at expansion joints. Longitudinal stresses were computed to be, in general, quite low. The maximum in the penstock was only 4,000 lb per sq in. and in the Dick's Creek crossing only 6,500 lb per sq in. Strain gage measurements made in the field, with and without water in the pipe, indicated stresses in satisfactory agreement with computed stresses.

Every precaution was taken to insure good welding in the shop and in the field. Pipe was shop assembled in sections not exceeding 32 ft in length or 12 tons in weight. Longitudinal welds, which are the most highly stressed, were thus generally made in the shop (the exception being those in the surge tank) and were stress relieved and radiographed. Circumferential welds were stress relieved. Longitudinal shop welds were machine made in double V-butt joints. Circumferential shop welds were hand welded in double V-butt joints.

Pipe sections were shipped to the job with interior spreaders to preserve their shape. It is especially important that sections be kept round and true at the ends, where they butt up against adjacent sections for welding.

Circumferential field joints on pipe in the open were double V-butt welds. Welds on all plate $1\frac{1}{4}$ in. thick and greater were radiographed. Where the pipe shell was $1\frac{1}{2}$ in. thick or more, it was electrically heated to 300° F before welding. To radiograph a joint, radium was placed at the center line of the pipe, and photographic film was placed outside, and in contact with, the joint of which a picture was desired. Lead shields were hung inside the pipe on either side of the radium to protect workmen from the dangerous radiations.

On circumferential field joints in tunnel linings, backing-up strips on single V-butt joints were used, as they could only be welded from the inside of the pipe.

Interior surfaces of all pipe steel were given one coat of bitumastic solution, applied cold. Exterior surfaces were not shop painted. After steel had been placed and welded, interior surfaces were given an additional cold coat of bitumastic solution, and a final coat of bitumastic enamel was applied hot. Exterior surfaces against which concrete was to be placed were left unpainted and exterior surfaces exposed to the air were given one coat of an alkyd base red lead primer, followed by two coats of aluminum in an alkyd vehicle. After the development had been in operation 5 years, the paint was still "standing up" well.

Steel tunnel lining was placed in sections of various lengths, the maximum being 32 ft. The pipe sections were shop equipped with grout connections at the top and at the invert. Those at the top were spaced about 16 ft on centers, and 2-in.-diameter pipes were inserted to make a contact with rock. Those at the bottom were spaced about 10 ft on centers, and were for $\frac{3}{4}$-in. pipe. After concrete had been placed around the steel lining, and had set thoroughly, the 2-in. pipes at the top were grouted, under pressure rising gradually to 200 lb per sq in. Quite often the concrete placed under the invert does not make good contact with the steel, probably because of air bubbles, and the $\frac{3}{4}$-in. pipes were grouted at low pressure to insure contact. High pressure bulges the steel very quickly.

At each of the three places where steel tunnel linings ended, concrete lining was placed for 6 ft beyond the steel. Eight radial holes at 45° intervals

were drilled 28 ft into rock through this lining, and the holes grouted to 500 lb per sq in. When this grout had set, a second set of eight holes, intermediately spaced, was drilled and grouted to the same pressure. Thus, a barrier was created against seepage along the tunnel.

The steel tee connecting the surge tank riser and the tunnel lining was shop fabricated, with all welds stress relieved and radiographed. It is known that, under pressure, a tee will develop stress concentrations, and it was considered inadvisable to accentuate these by field welding.

The surge tank riser and the surge tank lining were grouted after concrete backfill had set to insure that no hollows existed which would permit the lining to "work" under variations in water level.

All pipe anchors were designed for a safety factor of 2 against overturning, and were dimensioned so that the force tending to slide the anchor was not more than 0.8 of its weight, considering the anchor to be resisting (by gravity alone) the most severe combination of forces that could be imposed. In addition, all but one were doweled into rock. One anchor was founded on a soft brownish shale and was not doweled; and the maximum foundation pressure was limited, in this case, to 2.5 tons per sq ft.

Surge tank and penstock designs were based on the assumption that a turbine discharge of 700 cu ft per sec would be shut off in 5 sec. The effect of water hammer was computed by the arithmetic integration method introduced by Norman R. Gibson,[5] M. ASCE. Computations for surge were also made by a method of arithmetic integration. Efficiency tests were run on the turbine in August, 1942, using the Gibson method, the action of the surge tank being observed at the same time. Results indicated that the action is in accordance with design assumptions, and may be predicted for any operating condition with a reasonable degree of accuracy.

Observations of friction head loss at all rates of flow indicate that Kutter's n has values of 0.011 for lined tunnel or steel pipe, and 0.0295 for unlined tunnel, respectively, as determined by the Manning formula.

Acknowledgments

The Nantahala Hydroelectric Plant is owned by the Nantahala Power and Light Company, a subsidiary of the Aluminum Company of America. It was designed and built under the supervision of J. P. Growdon, M. ASCE, chief hydraulic engineer of the Aluminum Company of America, the writer being then principal assistant hydraulic engineer. The general contractor was the Utah Construction Company. All steel pipe was furnished and erected by the Chicago Bridge & Iron Company, and all field paint was applied by the Wailes Dove-Hermiston Corporation.

The design of the pipe line, penstock, and surge tank was the particular province of Frederick W. Ely, M. ASCE, and the writer wishes to acknowledge indebtedness to him for many of the data required in the paper.

[5] "Pressures in Penstocks Caused by the Gradual Closing of Turbine Gates," by Norman R. Gibson, *Transactions*, ASCE, Vol. LXXXIII, 1919–1920, p. 707.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

NUMERICAL COMPUTATION OF BUCKLING LOADS BY FINITE DIFFERENCES

By Mario G. Salvadori,[1] Assoc. M. ASCE

Synopsis

After introducing the concept of central differences of a function to approximate its derivatives by simple numerical expressions, the writer shows how a problem involving a linear differential equation of the homogeneous type, with homogeneous boundary conditions, may be reduced to a problem involving the solution of a system of simultaneous linear algebraic equations. In this manner solutions of buckling problems may be obtained by purely numerical computations, using a procedure of successive approximations, which is greatly enhanced by a simple method of extrapolation. The errors involved in the procedure are discussed and complete tables are given to be used in connection with extrapolation. The procedure is then applied to various cases of buckling of beams, plates, and shells to demonstrate the technique of its application and to solve a few problems not appearing in the literature.

1. INTRODUCTION

The evaluation of buckling loads of beams and plates often presents a difficult analytical problem, whose rigorous solution may become practically impossible when the shape of the plate is irregular or when the variation of the moment of inertia of the beam or plate does not follow a simple law.

For these reasons, approximate methods of computation have been devised, the best known among them perhaps being Rayleigh's energy method,[2] which gives upper bounds for the buckling loads. With the help of other methods giving lower bounds,[3,4,5] it is possible to squeeze the approximate load between given limits.

Note.—Written comments are invited for immediate publication; to insure publication, the last discussion should be submitted by May 1, 1950.

[1] Associate Prof. of Civ. Eng., Columbia Univ., New York, N. Y.

[2] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936.

[3] "Ein Gegenstück zum Ritzchen Verfahren," by E. Trefftz, *Proceedings*, 2d International Cong. for Applied Mechanics, Zürich, 1926.

[4] "On a Minimal Problem in Theory of Elasticity," by A. Weinstein, *Journal*, London Mathematical Soc., Vol. 10, 1935.

[5] "Étude des équations des plaques encastrées," by A. Weinstein, *Mémoires des Sciences Mathématiques*, Vol. 88, 1937.

Purely numerical methods for the solution of structural problems have become very popular among engineers, particularly since 1919, because of their simplicity and because of the wider availability of calculating machines. Hardy Cross,[6] Hon. M. ASCE, and R. V. Southwell[7,8] have pioneered in this field.

The numerical and graphical computation of buckling loads by successive approximations has been introduced by L. Vianello and A. Stodola, in connection with struts, and their method has been greatly improved by N. M. Newmark,[9] M. ASCE. Mr. Southwell[7,8] has applied his relaxation procedure to the same problem. Since 1910, however, L. F. Richardson[10] has suggested a finite difference procedure for the determination of characteristic values of differential equations, which was investigated independently by L. Collatz.[11] This procedure seems to present a very wide range of applications as well as great simplicity. It is the aim of this paper to give a comprehensive presentation of the finite difference method, together with some extrapolation procedures, greatly enhancing its usefulness.

2. CENTRAL DIFFERENCES

Let the values of a Taylor expansible function $y = f(x)$ be known at evenly spaced points of the x-axis of abscissas: $\cdots, x - 3h, x - 2h, x - h, x, x + h, x + 2h, x + 3h, \cdots$ (Fig. 1) and also, for the time being, at the middle point

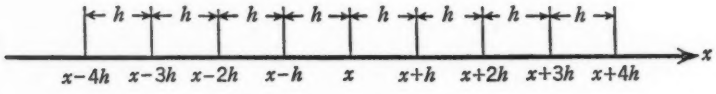

FIG. 1

of the intervals of width h so defined. The "first central difference" of $f(x)$ at x, indicated by the operational symbol δf (first introduced by W. F. Sheppard[12]) is defined by

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

The "second difference" of $f(x)$ at x is defined as the difference of the first difference and is indicated by the symbol $\delta^2 f(x)$:

$$\delta^2 f(x) = \delta[\delta f(x)] = f(x + h) - 2f(x) + f(x - h) \cdots\cdots\cdots (2)$$

[6] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932.

[7] "Relaxation Methods in Engineering Science," by R. V. Southwell, Clarendon Press, Oxford, England, 1940.

[8] "Relaxation Methods in Theoretical Physics," by R. V. Southwell, Clarendon Press, Oxford, England, 1947.

[9] "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," by N. M. Newmark, *Transactions*, ASCE, Vol. 108, 1943.

[10] "The Approximate Arithmetical Solution by Finite Differences of Physical Problems Involving Differential Equations with an Application to the Stresses in a Masonry Dam," by L. F. Richardson, *Philosophical Transactions*, Royal Soc. of London, Vol. 210, 1911.

[11] "Zusammenfassender Bericht: Genäherte Berechnung von Eigenwerten," by L. Collatz, *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 19, 1939.

[12] "Central Differences Formulae," by W. F. Sheppard, *Proceedings*, London Mathematical Soc., Vol. 31, 1899.

"Higher order differences" are defined by the recurrence equation:

$$\delta^n f(x) = \delta[\delta^{n-1} f(x)] \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (3)$$

which leads to

$$\delta^3 f(x) = f\left(x + \frac{3h}{2}\right) - 3f\left(x + \frac{h}{2}\right) + 3f\left(x - \frac{h}{2}\right) - f\left(x - \frac{3h}{2}\right) \quad (4a)$$

and

$$\delta^4 f(x) = f(x + 2h) - 4f(x + h) + 6f(x) - 4f(x - h) + f(x - 2h) \quad (4b)$$

The numerical coefficients of $\delta^n f(x)$ are the coefficients of the binomial expansion of $(a - b)^n$, and even order differences involve the values of $f(x \pm i\,h)$ $(i = 0, 1, 2, \cdots)$, whereas odd order differences require the knowledge of $f(x \pm i\,h/2)$. To eliminate these last values of $f(x)$, Mr. Sheppard[12] introduced a new operator μ, called the "averager" and defined by the equation:

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right] \quad \cdots\cdots\cdots\cdots\cdots (5a)$$

whose "square" is

$$\mu^2 f(x) = \mu[\mu f(x)] = \frac{1}{4}[f(x + h) + 2f(x) + f(x - h)] \quad \cdots\cdots (5b)$$

and whose nth power—

$$\mu^n f(x) = \mu[\mu^{n-1} f(x)] \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5c)$$

—has numerical coefficients identical with the coefficients of the binomial expansion of $(a + b)^n$ divided by 2^n. Comparison of Eqs. 2 and 5*b* shows that $\mu^2 f(x) - \frac{1}{4}\delta^2 f(x) = f(x)$, or, writing this equation in purely operational form by dropping the symbol $f(x)$ on both of its sides, that

$$\mu^2 = 1 + \frac{1}{4}\delta^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (6)$$

The successive derivatives of the function $f(x)$ are represented subsequently by the differential operator D:

$$D f(x) = \frac{df}{dx} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7a)$$

whose nth "power" is

$$D^n f(x) = D[D^{n-1} f(x)] = \frac{d^n f}{dx^n} \quad \cdots\cdots\cdots\cdots\cdots (7b)$$

It may be easily shown that the three operators δ, μ, and D satisfy, formally, the fundamental laws of algebra and that hence they can be used as if they were algebraic quantities, provided that the result of the operations is interpretable in terms of their definition.

Thus, the "products" $\mu\,\delta$ and $\mu\,\delta^3$ are defined as

$$\mu\,\delta[f(x)] = \mu[\delta f(x)] = \frac{1}{2}[f(x + h) - f(x - h)] \quad \cdots\cdots\cdots (8a)$$

and

$$\mu\,\delta^2[f(x)] = \mu[\delta^3 f(x)]$$
$$= \frac{1}{2}[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)] \quad \ldots\ldots (8b)$$

and the product $\mu\,\delta^{2m+1} f(x)$ $(m = 0, 1, 2, \cdots)$ involves only the values of $f(x)$ at $x + ih$ $(i = 0, 1, 2, \cdots)$.

Some important relationships between the operators δ, μ, and D may be easily obtained from the Taylor expansion of $f(x)$:

$$f(x+h) = f(x) + \frac{h}{1!}Df(x) + \frac{h^2}{2!}D^2 f(x) + \frac{h^3}{3!}D^3 f(x) + \cdots$$

which, "factoring" the function $f(x)$ and remembering the power series expansion of e^x, may be written symbolically as

$$f(x+h) = e^{hD} f(x) \quad \ldots\ldots (9a)$$

Changing h into $-h$, Eq. 9*a* gives

$$f(x-h) = e^{-hD} f(x) \quad \ldots\ldots (9b)$$

and, by Eqs. 9, Eq. 8*a* becomes: $\mu\,\delta f(x) = \frac{1}{2}[f(x+h) - f(x-h)]$ $= \frac{1}{2}[e^{hD} f(x) - e^{-hD} f(x)] = \frac{e^{hD} - e^{-hD}}{2} f(x) = \sinh(hD) f(x)$; or, in purely operational form,

$$\mu\,\delta = \sinh(hD) \quad \ldots\ldots (10a)$$

Expanding $\sinh(hD)$ into a power series, the operator $\mu\,\delta$ can finally be represented by the following series involving all the derivatives of $f(x)$:

$$\mu\,\delta = hD + \frac{h^3 D^3}{3!} + \frac{h^5 D^5}{5!} + \cdots \quad \ldots\ldots (10b)$$

Similarly, by Eqs. 9, $\delta^2 f$ becomes $\delta^2 f(x) = 2\left(\frac{e^{hD} + e^{-hD}}{2} - 1\right) f(x)$; or, in symbolical form,

$$\delta^2 = 2[\cosh(hD) - 1] = h^2 D^2 + \frac{h^4 D^4}{12} + \frac{h^6 D^6}{360} + \cdots \quad \ldots\ldots (11)$$

Multiplying Eq. 10*b* by Eq. 11,

$$\mu\,\delta^3 = h^3 D^3 + \frac{h^5 D^5}{4} + \frac{h^7 D^7}{45} + \cdots \quad \ldots\ldots (12a)$$

and, squaring Eq. 11,

$$\delta^4 = h^4 D^4 + \frac{h^6 D^6}{6} + \frac{h^8 D^8}{80} + \cdots \quad \ldots\ldots (12b)$$

Thus, the operators $\frac{\delta^{2m}}{h^{2m}}$ and $\mu\frac{\delta^{2m+1}}{h^{2m+1}}$ (m being an integer) may be expressed, in terms of the derivatives of $f(x)$, by power series in h involving only the even powers of h.

The inverse relationships, giving the successive derivatives of $f(x)$ in terms of its differences, are obtained as follows: Solving Eq. 10*a* for $h\,D$ and remembering the power series expansion of $\sinh^{-1} x$,

$$h\,D = \sinh^{-1}\mu\,\delta = \mu\,\delta - \frac{\mu^3\,\delta^3}{6} + \frac{3\,\mu^5\,\delta^5}{40} - \cdots$$

or, making use of Eq. 6 to eliminate all the even powers of μ,

$$h\,D = \mu\left(\delta - \frac{\delta^3}{6} + \frac{\delta^5}{30} - \cdots\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Similarly, squaring Eq. 13 and again using Eq. 6,

$$h^2\,D^2 = \delta^2 - \frac{\delta^4}{12} + \frac{\delta^6}{90} - \cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots (14a)$$

multiplying Eq. 13 by Eq. 14*a*,

$$h^3\,D^3 = \mu\left(\delta^3 - \frac{\delta^5}{4} + \frac{7\,\delta^7}{120} - \cdots\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (14b)$$

and, squaring Eq. 14*a*,

$$h^4\,D^4 = \delta^4 - \frac{\delta^6}{6} + \frac{7\,\delta^8}{240} - \cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots (14c)$$

3. DIFFERENCE OPERATORS IN CARTESIAN COORDINATES

A. Ordinary Operators.—In the approximate solution of differential equations by finite differences it is often assumed that all the terms at the right-hand members of Eqs. 13 to 14*c* beyond the first are negligible and that, hence, the derivatives of $f(x)$ may be approximated by

$$h\,D\,f(x) \doteq \mu\,\delta\,f(x) = \frac{1}{2}[f(x+h) - f(x-h)] \quad \ldots\ldots\ldots (15a)$$

$$h^2\,D^2 f(x) \doteq \delta^2 f(x) = f(x+h) - 2\,f(x) + f(x-h) \quad \ldots\ldots\ldots (15b)$$

$$h^3\,D^3 f(x) \doteq \mu\,\delta^3 f(x) = \frac{1}{2}[f(x+2\,h) - 2\,f(x+h) + 2\,f(x-h) - f(x-2\,h)] \quad \ldots (15c)$$

$$h^4\,D^4 f(x) \doteq \delta^4 f(x) = f(x+2\,h) - 4\,f(x+h) + 6\,f(x) - 4\,f(x-h) + f(x-2\,h) \quad \ldots (15d)$$

in which $\doteq$ represents "approximately equal to." By Eqs. 10*b* to 12*b*, these relations imply that the higher derivatives of $f(x)$ are not large in comparison with its lower derivatives and that the higher powers of h are negligible in comparison with its lower powers. (The error inherent in this approximation is evaluated in Section 5.)

The right-hand members of Eqs. 15 are called the "first order central differences" of $f(x)$ and may be represented graphically by the convenient schemes of Fig. 2.

By taking into account two terms of the series of Eqs. 13 and 14 the derivatives of $f(x)$ may be represented in terms of "second order central differences," as follows:

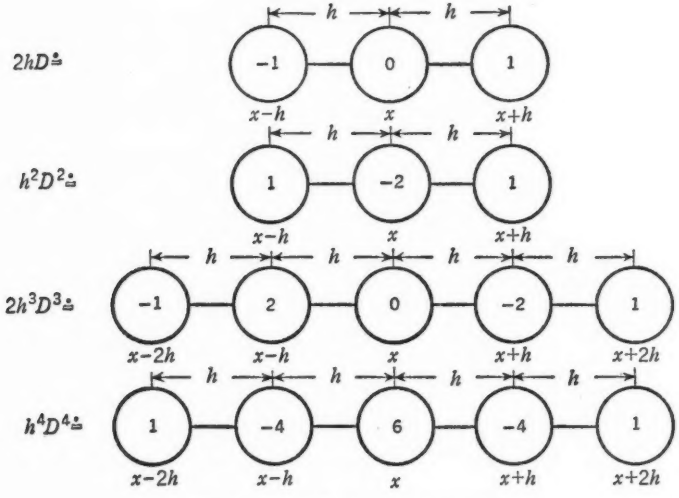

FIG. 2.—FIRST ORDER DIFFERENCES

$$h\,Df(x) \doteq \mu\left(\delta - \frac{\delta^3}{6}\right) f(x)$$

$$= \frac{1}{12}[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)] \ldots (16a)$$

$$h^2\,D^2 f(x) \doteq \left(\delta^2 - \frac{\delta^4}{12}\right) f(x) = \frac{1}{12}[-f(x+2h) + 16f(x+h)$$

$$- 30f(x) + 16f(x-h) - f(x-2h)] \ldots (16b)$$

$$h^3\,D^3 f(x) \doteq \mu\left(\delta^3 - \frac{\delta^5}{4}\right) f(x) = \frac{1}{8}[-f(x+3h) + 8f(x+2h)$$

$$- 13f(x+h) + 13f(x-2h) + f(x+3h)] \ldots (16c)$$

$$h^4\,D^4 \doteq \left(\delta^4 - \frac{\delta^6}{6}\right) f(x) = \frac{1}{6}[-f(x+3h) + 12f(x+2h) - 39f(x+h)$$

$$+ 56f(x) - 39f(x-h) + 12f(x-2h) - f(x-3h)] \ldots (16d)$$

The right-hand members of Eqs. 16 are represented graphically in Fig. 3.

Second order difference operators are more accurate than first order operators, but involve the values of $f(x)$ at a larger number of points. The

Nth order difference operators are similarly obtained. Since difference ratios approach the corresponding derivatives as h approaches zero, the accuracy of the numerical approximations may be increased either by making h smaller or by taking higher order differences. Which procedure is more advantageous depends on the type of problem at hand, but it is seldom that the additional labor required by higher order differences is warranted in practice by the results obtainable.

(Expressions for the derivatives in terms of differences may also be obtained by passing nth order parabolas through the points of abscissa $x \pm i\,h$ and of ordinate $y = f(x \pm i\,h)$, and by substituting the derivatives of the interpolated parabola for the derivatives of $f(x)$. When the interpolation points are symmetrically located with respect to x, the formulas thus obtained are identical with the central difference operators given in Eqs. 15 and 16. When the interpolation points are not symmetrically located with respect to x, the formulas give the derivatives in terms of "lateral differences." This latter type of formula is conveniently used on the boundaries of beams and plates, but will not be considered in this paper. Mr. Southwell lists[8] most of the central and lateral difference formulas needed in the applications, under the name of "Bickley's formulas.")

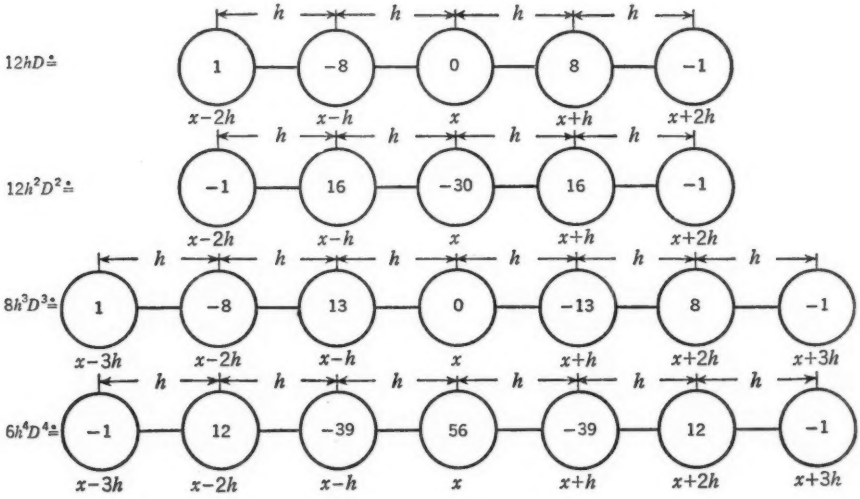

FIG. 3.—SECOND ORDER DIFFERENCES

B. Partial Operators.—The first order and second order difference expressions, Eqs. 15 and 16, may be used to evaluate the partial derivatives of a function $z = f(x,y)$ with respect to x (or y) by substituting for $D = \frac{d}{dx}$ the partial derivative symbol $D_x = \frac{\partial}{\partial x}$ $\left(\text{or } D_y = \frac{\partial}{\partial y}\right)$ and by holding y (or x) constant in computing the differences. Thus, using first order differences

and calling $r\,h$ the spacing of points in the y-direction,

$$h\,D_x\,z \doteq \frac{1}{2}[f(x+h,y) - f(x-h,y)] \quad \ldots\ldots\ldots\ldots (17a)$$

and

$$r\,h\,D_y\,z \doteq \frac{1}{2}[f(x,y+r\,h) - f(x,y-r\,h)] \quad \ldots\ldots\ldots\ldots (17b)$$

whereas the second mixed derivative of z with respect to x and y, $\frac{\partial^2 z}{\partial x\,\partial y} = D_{xy}\,f(x,y)$, is obtained by applying consecutively the operators at the right-hand members of Eqs. 17:

$$r\,h^2\,D_{xy}\,z \doteq \frac{1}{4}[f(x+h,\,y+r\,h) - f(x+h,\,y-r\,h) - f(x-h,\,y+r\,h) + f(x-h,\,y-r\,h)] \ldots (18a)$$

and the fourth mixed derivative $\frac{\partial^4 f}{\partial x^2\,\partial y^2}$ by squaring the operator D_{xy}:

$$r^2\,h^4\,D^2_{xy}\,z \doteq \frac{1}{16}\{f(x+h,\,y+r\,h) + f(x-h,\,y+r\,h) + f(x+h,\,y-r\,h) + f(x-h,\,y-r\,h) - 2[f(x,\,y+r\,h) + f(x+h,\,y) + f(x-h,\,y) + f(x,\,y-r\,h)] + 4\,f(x,\,y)\} \ldots (18b)$$

Similarly, the first order difference operator corresponding to the Laplacian $\nabla^2 z = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ becomes

$$h^2\,\nabla^2 z \doteq f(x+h,\,y) + f(x-h,\,y) + \left(\frac{1}{r^2}\right)[f(x,\,y+r\,h) + f(x,\,y-r\,h)] - 2\left(1+\frac{1}{r^2}\right) f(x,\,y) \ldots (19a)$$

and, in particular for $r = 1$,

$$h^2\,\nabla^2 z \doteq f(x+h,y) + f(x-h,y) + f(x,y+h) + f(x,y-h) - 4\,f(x,y) \ldots (19b)$$

Squaring Eq. 19*b*, the first order operator corresponding to $\nabla^4 z = \frac{\partial^4 f}{\partial x^4} + 2\,\frac{\partial^4 f}{\partial x^2\,\partial y^2} + \frac{\partial^4 f}{\partial y^4}$ for the case $r = 1$ is found to be

$$h^4\,\nabla^4 z \doteq [f(x+2\,h,y) + f(x,y+2\,h) + f(x-2\,h,y) + f(x,y-2\,h)] + 2\,[f(x+h,\,y+h) + f(x-h,\,y+h) + f(x+h,\,y-h) + f(x-h,\,y-h)] - 8\,[f(x+h,y) + f(x-h,y) + f(x,y-h) + f(x,y+h)] + 20\,f(x,y) \ldots (19c)$$

Using second order differences (Eqs. 16) the expressions for second order partial difference operators may be similarly obtained.

The schemes of Fig. 4 and Fig. 5 give some of the first order and second order operators most commonly encountered, showing how partial difference operators involve the values of the function z at the corners of a rectangular or square grid. Partial operators in Cartesian coordinates are well adapted to the solution of problems involving rectangular or square domains.

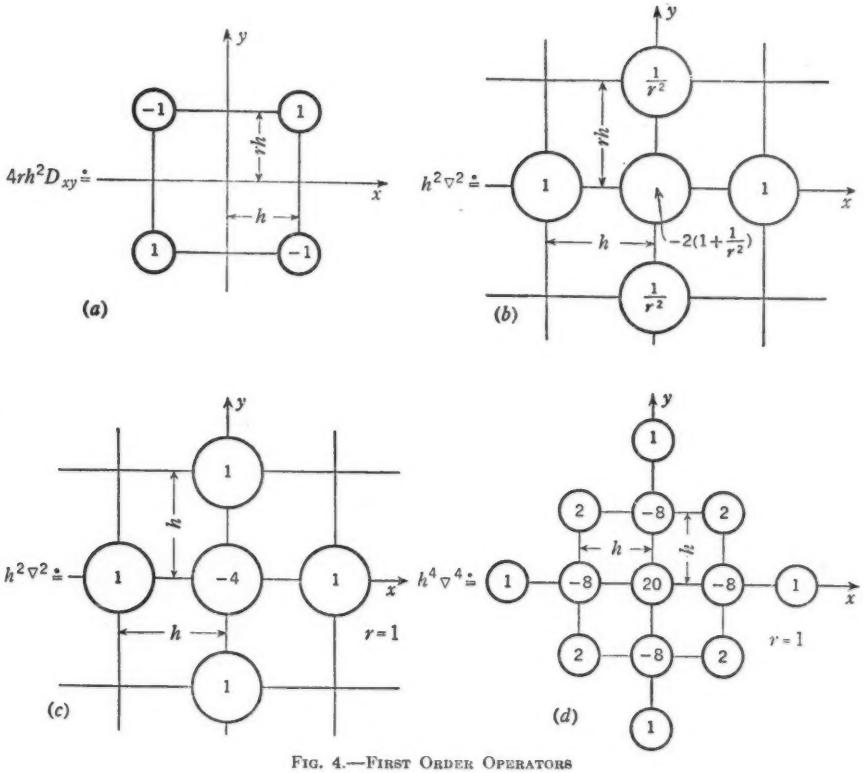

FIG. 4.—FIRST ORDER OPERATORS

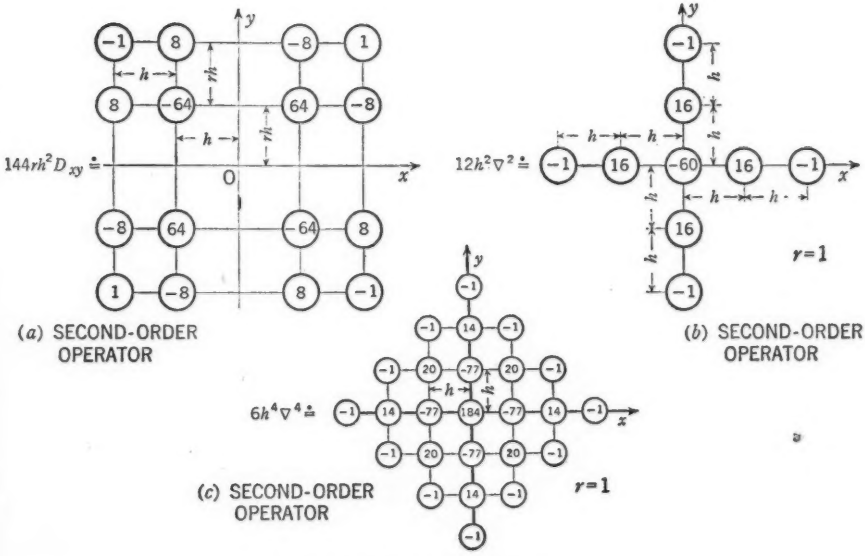

FIG. 5.—SECOND ORDER OPERATORS

4. DIFFERENCE OPERATORS IN SKEW COORDINATES

The solution of problems involving rhombic domains (for instance, skew plate problems) is obtained more conveniently by oblique Cartesian, or "skew," coordinates.[13,14]

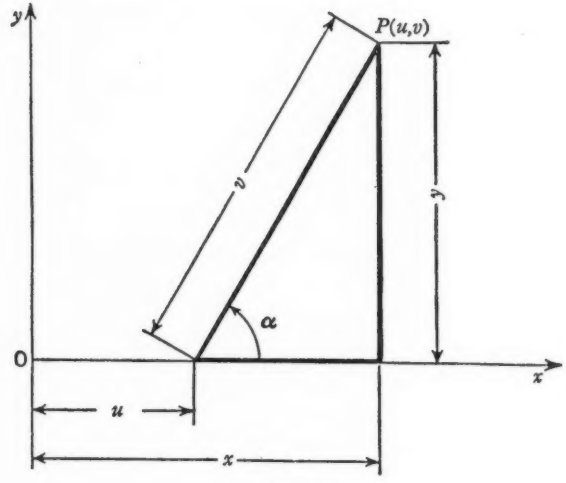

FIG. 6

Fig. 6 indicates the relationship between the Cartesian coordinates x,y of a point P and its skew coordinates u,v:

$$x = u + v\cos\alpha \qquad \text{and} \qquad y = v\sin\alpha \ldots\ldots\ldots\ldots (20a)$$

(in which α is the angle of skew) from which it is easy to derive the inverse relationships:

$$u = x - \frac{y}{\tan\alpha} \qquad \text{and} \qquad v = \frac{y}{\sin\alpha} \ldots\ldots\ldots\ldots (20b)$$

Given a function $z = f(x,y)$, its derivatives with respect to x and y may be written in terms of $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using Eq. 20*b*:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \ldots\ldots\ldots\ldots\ldots\ldots (21a)$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -\frac{1}{\tan\alpha}\frac{\partial z}{\partial u} + \frac{1}{\sin\alpha}\frac{\partial z}{\partial v} \ldots\ldots\ldots (21b)$$

[13] "Le calcul des plaques obliques par la méthode des équations aux differences," by H. Favre, *Mémoires*, Association Internationale de Ponts et Charpentes, Vol. 7, Zürich, 1943–1944.

[14] "Theoretische Untersuchungen über die Eigenfrequenz parallelogramförmiger Platten," by R. Bereuter, Publication du Laboratoire de Photo-Élasticité de l'École Polytechnique Fédérale, Zürich, 1946

The expressions for the higher derivatives of z are obtained by repeated application of Eqs. 21:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22a)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{\sin^2 \alpha}\left(\cos^2 \alpha \frac{\partial^2 z}{\partial u^2} - 2 \cos \alpha \frac{\partial^2 z}{\partial u\, \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \ldots\ldots\ldots (22b)$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{-1}{\tan \alpha} \frac{\partial^2 z}{\partial u^2} + \frac{1}{\sin \alpha} \frac{\partial^2 z}{\partial u\, \partial v} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22c)$$

$$\nabla^2 z = \frac{1}{\sin^2 \alpha}\left(\frac{\partial^2 z}{\partial u^2} - 2 \cos \alpha \frac{\partial^2 z}{\partial u\, \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \ldots\ldots\ldots\ldots (22d)$$

The corresponding difference operators are derived by substitution of central difference ratios taken in the u-direction and the v-direction for the derivatives in the same directions, according to Eqs. 15 or 16, depending upon

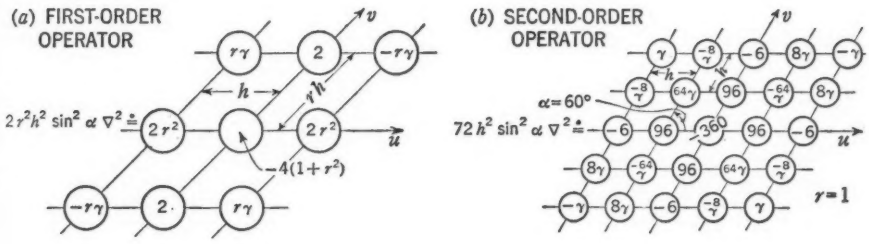

FIG. 7

whether first order or second order operators are required. Thus, letting h and $r\,h$ be the spacing of the points in the u-direction and the v-direction, then, with first order differences,

$$\begin{aligned} 2\,r^2\,h^2 \sin^2 \alpha\, \nabla^2 z \doteq\ & 2\,r^2\,[f(u+h,v) + f(u-h,v)] \\ & + 2\,[f(u, v+r\,h) + f(u, v-r\,h)] - 4\,(1+r^2)\,f(u,v) \\ & - r \cos \alpha\,[f(u+h, v+r\,h) + f(u-h, v-r\,h) \\ & \qquad - f(u+h, v-r\,h) - f(u-h, v+r\,h)] \quad \ldots (23) \end{aligned}$$

which is represented in Fig. 7, where $\gamma = \cos \alpha$.

5. DIFFERENCE OPERATORS IN TRIANGULAR COORDINATES

When the problem to be solved involves a domain coverable by an equilateral triangular grid (equilateral triangular plate, regular hexagonal plate, etc.), it is convenient to use the "equitriangular coordinates" shown in Fig. 8, in which a point is located by coordinates measured along three axes (x, u, and v) making 60° angles between them.

To obtain the operator ∇^2 of $z = f(x,y)$ in this new set of coordinates, notice that: $\frac{\partial x}{\partial u} = -\frac{\partial x}{\partial v} = \cos 60° = \frac{1}{2}$ and $\frac{\partial y}{\partial u} = \frac{\partial y}{\partial v} = \sin 60° = \frac{\sqrt{3}}{2}$. There-

fore:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{1}{2}\frac{\partial z}{\partial x} + \frac{\sqrt{3}}{2}\frac{\partial z}{\partial y} \quad \ldots\ldots\ldots\ldots (24a)$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\frac{1}{2}\frac{\partial z}{\partial x} + \frac{\sqrt{3}}{2}\frac{\partial z}{\partial y} \quad \ldots\ldots\ldots\ldots (24b)$$

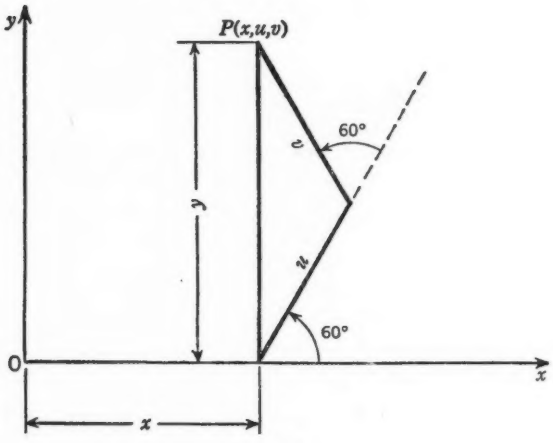

FIG. 8

Squaring Eqs. 24, the second derivatives of z with respect to u and v become

$$\frac{\partial^2 z}{\partial u^2} = \frac{1}{4}\frac{\partial^2 z}{\partial x^2} + \frac{3}{4}\frac{\partial^2 z}{\partial y^2} + \frac{\sqrt{3}}{2}\frac{\partial^2 z}{\partial x\,\partial y} \quad \ldots\ldots\ldots\ldots (25a)$$

and

$$\frac{\partial^2 z}{\partial v^2} = \frac{1}{4}\frac{\partial^2 z}{\partial x^2} + \frac{3}{4}\frac{\partial^2 z}{\partial y^2} - \frac{\sqrt{3}}{2}\frac{\partial^2 z}{\partial x\,\partial y} \quad \ldots\ldots\ldots\ldots (25b)$$

from which, by summation, $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{1}{2}\frac{\partial^2 z}{\partial x^2} + \frac{3}{2}\frac{\partial^2 z}{\partial y^2}$. Adding $\frac{\partial^2 z}{\partial x^2}$ to both sides of the latter equation, the expression for $\nabla^2 z$, in equitriangular coordinates, is

$$\nabla^2 z = \frac{2}{3}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) \quad \ldots\ldots\ldots\ldots (26)$$

To derive the corresponding difference operator, substitute in Eqs. 26 for the derivatives in the x-direction, the u-direction, and the v-direction the corresponding difference ratios in the same directions. If h is the equal spacing of the corners of the grid in the three directions, using first order differences,

$$h^2\,\nabla^2 z \doteq \tfrac{2}{3}\,[f(x+h,u,v) + f(x-h,u,v) + f(x,u+h,v) + f(x,u-h,v) + f(x,u,v+h) + f(x,u,v-h) - 6\,f(x,u,v)] \ldots (27)$$

The graphs of Fig. 9 give the expressions of $\nabla^2 z$ by first order and second order differences and of $\nabla^4 z$ by first order differences.

V. P. Jensen,[15] M. ASCE, has given the expressions for $\nabla^2 z$ and $\nabla^4 z$ in "nonequiangular triangular coordinates," by which finite difference operators

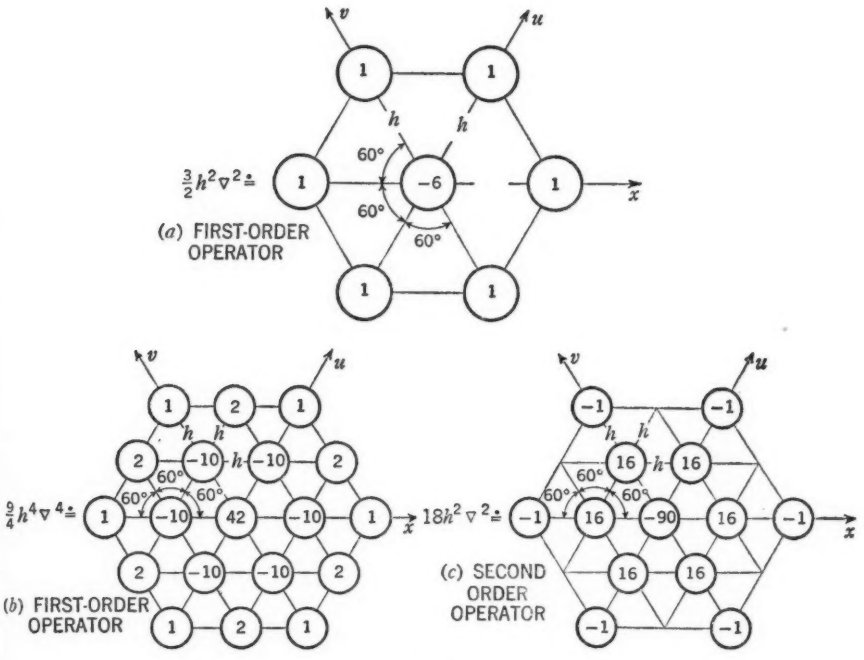

FIG. 9

can be applied to domains coverable by a grid of equal irregular triangles. Mr. Jensen's finite difference operators are first order operators.

6. ERROR IN THE DERIVATIVES DUE TO THE SUBSTITUTION OF DIFFERENCE RATIOS

As noted in Section 3, in the solution of differential equations by finite differences the derivatives of the unknown function are approximated by the corresponding difference ratios.

When first order differences are used, the error e'_1 in the value of the first derivative is immediately found from Eq. 10*b*:

$$e'_1 = \left(D - \frac{\mu\,\delta}{h}\right) f(x) = -\frac{D^3 f}{3!}h^2 - \frac{D^5 f}{5!}h^4 - \cdots$$

[15] "Analyses of Skew Slabs," by V. P. Jensen, *Bulletin Series No. 332*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1941.

Similarly, the error in the second derivative becomes by Eq. 11:

$$e'_2 = \left(D^2 - \frac{\delta^2}{h^2} \right) f(x) = -\frac{D^4 f}{12} h^2 - \frac{D^6 f}{360} h^4 - \cdots$$

and, more generally, the error in the nth derivative is of the type:

$$e'_n = \phi'_1(x)\, h^2 + \phi'_2(x)\, h^4 + \phi'_3(x)\, h^6 + \cdots \cdots \cdots \cdots \cdots (28)$$

in which the functions $\phi'_i(x)$ are independent of h.

Eq. 28 shows that the error due to first order differences depends on the even powers of h only and that, when the Taylor series of $f(x)$ is converging rapidly (small values of h and rapidly decreasing values of the successive derivatives of $f(x)$), the series at its right-hand member will also converge rapidly—the more so as n becomes greater.

To evaluate the error due to second order differences, note that by Eq. 16*a* the first derivative is approximated by the difference ratio:

$$D f(x) \doteq \left(\frac{\mu\, \delta}{h} - \frac{\mu\, \delta^3}{6\, h} \right) f(x).$$

Substituting in this latter equation the expression $D f$ obtained from Eq. 10*b* and the expression $\mu\, \delta^3$ obtained from Eq. 12*a*, the error e''_1 becomes

$$e''_1 = \left(D - \frac{\mu\, \delta}{h} + \frac{\mu\, \delta^3}{6\, h} \right) f(x) = \frac{D^5 f}{30} h^4 + \frac{159\, D^7 f}{45{,}360} h^6 + \cdots$$

Similarly, by Eqs. 16*b*, 11, and 12*b*, the error in the second derivative is

$$e''_2 = \left(D^2 - \frac{\delta^2}{h^2} + \frac{\delta^4}{12\, h^2} \right) f(x) = \frac{23\, D^6 f}{2{,}520} h^4 + \frac{D^8 f}{21} h^6 + \cdots$$

and, more generally, the error in the nth derivative is of the type:

$$e''_n = \phi''_1(x)\, h^4 + \phi''_2(x)\, h^6 + \phi''_3(x)\, h^8 + \cdots \cdots \cdots \cdots (29a)$$

When the Taylor series of $f(x)$ is rapidly convergent, the series on the right-hand member of Eq. 29*a* is also rapidly convergent, and more so than the series of Eq. 28.

The derivations of this section show that the error due to the approximation of the nth derivative by mth order difference ratios is of the type:

$$e_{nm} = \phi_{1m}\, h^{2m} + \phi_{2m}\, h^{2(m+1)} + \phi_{3m}\, h^{2(m+2)} + \cdots \cdots \cdots \cdots (29b)$$

in which the functions ϕ_{im} are independent of h.

On the basis of this result (due to Mr. Sheppard[12]) Mr. Richardson[10] was able to prove that the error in the solution of a problem defined by a linear differential equation and linear boundary conditions, due to the substitution of difference ratios for derivatives, is also of the "h^{2m}-type," as expressed by Eq. 29*b*.

7. ERROR IN THE CHARACTERISTIC VALUES DUE TO THE SUBSTITUTION OF DIFFERENCE RATIOS FOR DERIVATIVES

The solution of buckling and vibration problems requires the determination of the values of a certain parameter K for which the linear homogeneous differential equation of the problem has solutions, satisfying linear homogeneous boundary conditions, and not identically equal to zero.

Calling Y one such solution and indicating by $L_1 Y$ and $L_2 Y$ two linear combinations of derivatives of Y, a buckling equation, for instance, may be simply written as

$$L_1 Y + K L_2 Y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Next indicate by l_1 and l_2 the difference operators corresponding to L_1 and L_2; by y the approximate value of Y; by k the approximate value of K obtained by solving the difference rather than the differential problem; and by e the error of k, that is, the difference:

$$e = K - k \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

With these symbols the approximating difference equation becomes

$$l_1 y + k l_2 y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

Indicate by e_1 and e_2 the errors due to substitution of finite differences for derivatives in $L_1 y$ and $L_2 y$:

$$L_1 y - l_1 y = e_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (33a)$$

and

$$L_2 y - l_2 y = e_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (33b)$$

and substitute Eqs. 31 and 33 in Eq. 32, thus: $L_1 y - e_1 + (K - e)(L_2 y - e_2) = 0$, from which

$$e = \frac{(L_1 y + K L_2 y) - (e_1 + K e_2)}{L_2 y - e_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

In his classical paper on finite differences applied to the solution of differential equations, Mr. Richardson[10] makes the assumption that the error e, of K, is also of the h^{2m}-type. Eq. 34 proves this assumption generally untrue. In fact, by Mr. Richardson's theorem e_1 and e_2 (and hence $e_1 + K e_2$) are of the h^{2m}-type, whereas in general, $L_1 y + K L_2 y$ is not. Therefore, performing the division indicated in Eq. 34 and assuming, for instance, first order difference operators, e takes the form:

$$e = \phi_0(h) + \phi_1(h)\, h^2 + \phi_2(h)\, h^4 + \cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

which is not of the h^2 type, both because of the first term ϕ_0 and because of the dependence of the other ϕ_i on h. (This dependence is due to the fact that e_1 and e_2 are obviously functions of the spacing h.)

8. THE EVALUATION OF BUCKLING LOADS BY FINITE DIFFERENCES

When the differential equation—

$$L_1 f + K L_2 f = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

—of a buckling problem is known, together with the corresponding boundary conditions $L_3 f = 0$, the characteristic values K may be determined by the following method of successive approximations, based on the use of finite differences:

The domain under consideration is covered by one of the grids considered in Sections 3, 4, and 5, some of whose corners will fall on the boundary of the domain (boundary points) and some of which will fall inside the boundary (body points). The values of f at the boundary points and at some neighboring points will be determined by the boundary conditions of the problem, directly or in terms of the body points, as will be noted in detail from the applications of Section 10. The values f_i of f at the N body points of the mesh ($i = 1, 2, \cdots, N$) are unknown instead.

The difference equation—

$$l_1 f + k\, l_2 f = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37a)$$

—corresponding to the differential Eq. 36 is satisfied at each body point. Hence the application of Eq. 37*a* at each of the N body points gives rise to a set of N linear, algebraic, homogeneous equations in the N unknowns f_i:

$$l_1 f_i + k\, l_2 f_i = 0 \quad (i = 1, 2, \cdots, N) \quad \ldots\ldots\ldots\ldots\ldots (37b)$$

and this system has solutions f_i not identically zero, if and only if its determinant Δ, which is a function of k, is equal to zero:

$$\Delta(k) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

If p is the degree in k of the determinantal Eq. 38 (k-equation), its p roots give approximate values of the first p characteristic values, K, of the original differential equation, Eq. 36. In problems of elastic stability the designer will be mainly interested in the lowest buckling load, that is, in the lowest characteristic value, and will hence solve Eq. 38 for its smallest root k. Once this is done for a value h_i of the mesh size h, obtaining a value k_i for k, the computations may be repeated for a new mesh size h_j smaller than h_i, to obtain a better approximation k_j of k. By reduction of the mesh size and corresponding increase in the number N of the body points considered, K may be evaluated, theoretically, to any degree of accuracy.

Instead, an increase in the number of body points actually increases very rapidly the labor involved in writing out and solving the determinantal equation and it becomes practically impossible to better the accuracy of the solution beyond certain limits. For this reason it is important to use a practical method of evaluation for the determinantal equation and a simple procedure for its solution.

Once the determinant has been obtained by the schemes of Sections 3, 4, and 5, the simplest procedure for writing the determinantal equation is by the method of "pivotal condensation." It can be easily proved[16] that the value of a determinant of order n—

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (39a)$$

—is equal to the value of the following determinant of order $n - 1$:

$$\Delta = \frac{1}{a^{n-2}{}_{11}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix} \quad \ldots\ldots\ldots (39b)$$

in which each element is a second order determinant with the same leading element a_{11}, called the "pivot." Any other element a_{ij} of Δ may be used as pivot, but in this case the determinant of Eq. 39*b* must be given the sign $(-1)^{i+j}$.

By successive applications of Eq. 39*b* to Eq. 38 the order of the determinant $\Delta(k)$ is reduced to 1 and the k-equation is thus obtained. In this evaluation the multiplying factor $\dfrac{(-1)^{i+j}}{a^{n-2}{}_{ij}}$ may be dropped at each step, since the determinant must be equated to zero. All the elements of any row or column may also be divided by a common factor for the same reason, thus simplifying the numerical coefficients of the final k-equation.

When $\Delta(k)$ is a determinant of high order, each term of the successive determinants obtained by pivotal condensation is a high degree polynomial in k and writing out the k-equation may become cumbersome. It is easier, in such cases, to substitute an approximate numerical value of k in $\Delta(k)$ and to evaluate by pivotal condensation the corresponding numerical determinant. Two or three trials will usually give an acceptable result after interpolation.

The most practical method for the determination of the lowest root of the determinantal equation is Newton's method or the method of the tangents,[17] in which k_i is used as a starting value for the computation of k_j $(j > i)$.

9. EXTRAPOLATION PROCEDURES FOR THE EVALUATION OF BUCKLING LOADS

As shown in Section 8, the approximate values k_i approach the true value of K as the mesh size h_i approaches zero. The mode of this approach depends

[16] "Determinants and Matrices," by A. C. Aitken, Oliver and Boyd, London, 1939, p. 45.

[17] "Higher Mathematics for Engineers and Physicists," by I. E. Sokolnikoff and E. S. Sokolnikoff, McGraw-Hill Book Co., Inc., New York, N. Y, 1st Ed., 1934, p. 32.

on the particular problem at hand—that is, on the differential equation and the boundary conditions, and it is not known, offhand.

On the basis of extensive experimental investigations on the numerical evaluation of buckling loads by finite differences, the following qualitative statements may be made as to the behavior of k:

A. In many buckling problems, for which exact solutions are known, the approximations (k) of K approach the correct value monotonically (Fig. 10(a)) —that is, either continuously decreasing or continuously increasing.

B. In some cases the approach of k to K is not monotonic for large values of h, but becomes rapidly monotonic as h approaches zero (Fig. 10(b)).

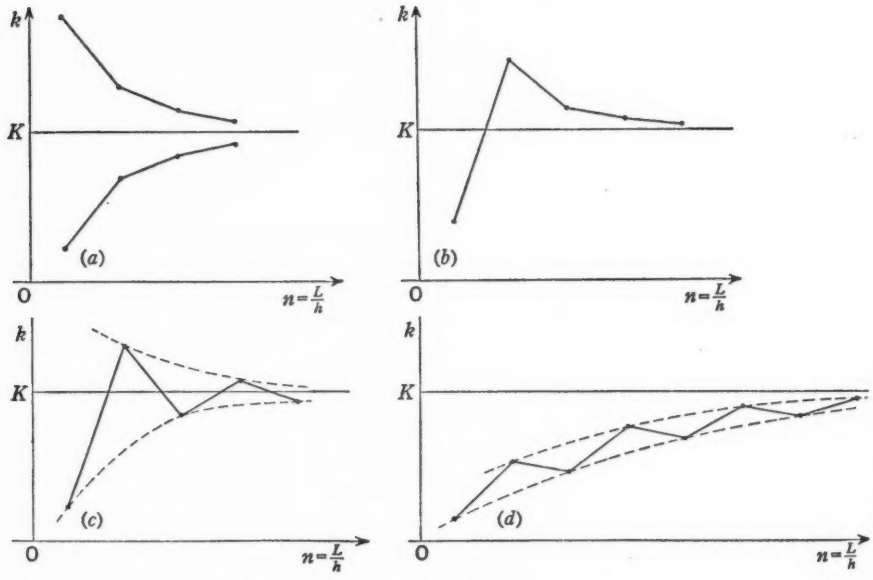

FIG. 10.—APPROACH OF k TO K

C. In a substantial number of cases the approach of k to K is oscillatory with decreasing amplitude (Fig. 10(c))—that is, k approaches K alternatively from above and from below.

D. In some cases the approach of k to K is one sided (from above or below), but not monotonic (Fig. 10(d)).

E. No other kinds of approach have been encountered thus far.

F. By choosing a proper sequence of values for h it is possible in all cases to obtain a monotonic sequence of values of k.

G. The error corresponding to a monotonic sequence is found, empirically, to be of the h^{2m}-type—that is, for a monotonic sequence the functions $\phi_{im}(h)$ of Eq. 35 are practically independent of h, and ϕ_0 is practically zero. The

error e', due to first order differences, may thus be written as

$$e' = a\,h^2 + b\,h^4 + c\,h^6 + \cdots \quad (40a)$$

in which a, b, c, $\cdots$ are constants.

This last result makes it possible to evaluate the characteristic value by extrapolation procedures, first suggested by Mr. Richardson,[10] which enhance the powerfulness of the method to a considerable extent.

By Eq. 40*a*, when h is sufficiently small for its higher powers to be negligible in comparison with h^2, e' is approximately given by.

$$e' \doteq a\,h^2 \quad (40b)$$

If n_i and n_j represent the number of subintervals in which a characteristic dimension L of the domain is divided, corresponding to the mesh sizes $h_i = L/n_i$ and $h_j = L/n_j$, and if k_i and k_j are the corresponding approximate values of K and e'_i, e'_j are the corresponding errors, Eq. 40*b* gives: $e'_i = K - k_i \doteq \frac{a\,L^2}{n^2_i}$ and $e'_j = K - k_j \doteq \frac{a\,L^2}{n^2_j}$. Eliminating $a\,L^2$ between these two equations, the "h^2-extrapolation formula" is obtained:

$$k'_{i,j} = \frac{n^2_j\,k_j - n^2_i\,k_i}{n^2_j - n^2_i} = a_j\,k_j - a_i\,k_i \quad (41)$$

in which $k'_{i,j}$ is the extrapolated value of K. Table 1 gives the coefficients a_j, a_i of the h^2-extrapolation formula for the most commonly used values of n_i and n_j.

TABLE 1.—h^2-EXTRAPOLATIONS

n_j/n_i	a_j		a_i	
(1)	(2)	(3)	(4)	(5)
2/1	4/3	1.33333	1/3	0.33333
3/2	9/5	1.8	4/5	0.8
4/3	16/7	2.28571	9/7	1.28571
5/4	25/9	2.77778	16/9	1.77778
5/3	25/16	1.5625	9/16	0.5625
6/5	36/11	3.272727	25/11	2.272727
7/5	49/24	2.041667	25/24	1.041667
7/6	49/13	3.769234	36/13	2.769234

Since Eq. 41 is homogeneous in n, the h^2-extrapolation coefficients of Table 1 are only functions of the ratios n_j/n_i; for instance, the (2/1)-extrapolation formula holds also for (4/2).

If, in Eq. 40*a*, the first two terms of the series are taken into consideration and the formula is applied to three approximations k_i, k_j, and k_k, obtained by subdividing the characteristic length L into $n_i < n_j < n_k$ parts, elimination

of $a\,L^2$ and $b\,L^2$ gives the "(h^2,h^4)-extrapolation formula":

$$k'_{i,j,k} = \frac{n^4_k(n^2_j - n^2_i)k_k - n^4_j(n^2_k - n^2_i)k_j + n^4_i(n^2_k - n^2_j)k_i}{n^4_k(n^2_j - n^2_i) - n^4_j(n^2_k - n^2_i) + n^4_i(n^2_k - n^2_j)}$$
$$= a_k\,k_k - a_j\,k_j + a_i\,k_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

in which k' is the extrapolated value of K.

Table 2 gives the (h^2,h^4)-extrapolation coefficients for the most commonly used values of n_i, n_j, and n_k. Again the values of n may be multiplied by a common factor without changing the formulas.

TABLE 2.—(h^2,h^4)-EXTRAPOLATIONS

n_i,n_j,n_k	a_k		a_j		a_i	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
3/2/1	243/120	2.025	128/120	1.06667	5/120	0.04167
4/3/2	1,280/420	3.04762	972/420	2.31429	112/420	0.26667
5/4/3	4,375/1,008	4.34028	4,096/1,008	4.06349	729/1,008	0.72321
6/5/4	11,664/1,980	5.89091	12,500/1,980	6.31313	2,816/1,980	1.42222
7/6/5	26,411/3,432	7.69551	31,104/3,432	9.06294	8,125/3,432	2.36742
5/3/1	5,000/3,072	1.62760	1,944/3,072	0.63281	16/3,072	0.00521
7/5/3	38,416/15,360	2.50104	25,000/15,360	1.62760	1,944/15,360	0.12656

The h^2-extrapolation formula and the (h^2,h^4)-extrapolation formula are used in connection with first order differences.

Making the same type of simplifying assumptions in the formula for the error e'' due to second order differences,

$$e'' = a\,h^4 + b\,h^6 + c\,h^8 + \cdots$$

the "h^4-extrapolation formula" is

$$k''_{i,j} = \frac{n^4_j\,k_j - n^4_i\,k_i}{n^4_j - n^4_i} = b_j\,k_j - b_i\,k_i \ldots\ldots\ldots\ldots\ldots (43a)$$

and the "(h^4,h^6)-extrapolation formula" is

$$k''_{i,j,k} = \frac{n^6_k(n^2_j - n^2_i)k_k - n^6_j(n^2_k - n^2_i)k_j + n^6_i(n^2_k - n^2_j)k_i}{n^6_k(n^2_j - n^2_i) - n^6_j(n^2_k - n^2_i) + n^6_i(n^2_k - n^2_j)}$$
$$= b_k\,k_k - b_j\,k_j + b_i\,k_i \ldots\ldots (43b)$$

Table 3 and Table 4 give the h^4-extrapolation coefficients and the (h^4,h^6)-extrapolation coefficients most often needed. All these formulas are homogeneous in n and hold when all the values of n are multiplied by the same number.

The following points should be kept in mind in using the extrapolation formulas of Section 9:

(*a*) The h^2-extrapolation formula and the h^4-extrapolation formula assume that the successive approximations k approach K from the same side. When the approach to K is of the oscillating type, the formulas may be used separately

in connection with the approximations from above or from below. The two extrapolated values thus obtained may be usefully compared. It is unsafe to use extrapolation without having at least three approximate values of K.

(*b*) Even if the approximate values of K approach the true value monotonically (either from above or from below), it is not known offhand whether the extrapolated value is above or below the true value.

TABLE 3.—h^4-EXTRAPOLATIONS

n_j/n_i	b_j		b_i	
(1)	(2)	(3)	(4)	(5)
2/1	16/15	1.06667	1/15	0.06667
3/2	81/65	1.24615	16/65	0.24615
4/3	256/175	1.46286	81/175	0.46286
5/4	625/369	1.69377	256/369	0.69377

TABLE 4.—(h^4,h^6)-EXTRAPOLATIONS

n_i,n_j,n_k	b_k		b_j		b_i	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
3/2/1	2,187/ 1,680	1.301786	512/ 1,680	0.304762	5/ 1,680	0.002976
4/3/2	20,480/12,180	1.681445	8,748/12,180	0.718227	448/12,180	0.036782
5/4/3	109,375/50,400	2.170139	65,536/50,400	1.300317	6,561/50,400	0.130179

(*c*) It is true that, in general, the (h^2,h^4)-extrapolation gives better results than the h^2-extrapolation, but it is impossible to determine offhand whether an h^2-extrapolation based on larger values of n will give better results than an (h^2,h^4)-extrapolation based on smaller values of n or even on three values of n the largest of which is the same as the largest value of n in the h^2-extrapolation. The same statement holds true for the h^4-extrapolation and the (h^4,h^6)-extrapolation.

10. APPLICATIONS

In this section the method of finite differences will be used to compute the lowest buckling load of beams, plates, and shells under various load and boundary conditions. The following applications have been chosen: (1) Because they illustrate the technical details and the properties of the method; (2) because they indicate the asymptotic approach of the successive approximations to the true value of the buckling load; or (3) because, to the best knowledge of the writer, the solution of the problem had not been previously attempted.

To simplify and shorten the treatment, the differential equations and boundary conditions of each problem, as well as the true or the best available approximate value of the buckling load, have been taken from the fundamental monograph by S. Timoshenko on "Theory of Elastic Stability"[2] unless otherwise stated.

A. Euler's Buckling Problem

A simply supported, slender beam, of length L and constant moment of inertia I, buckles under the action of two equal longitudinal loads P (Fig. 11(*a*)). Calling E Young's modulus and $y = f(x)$ the normal deflection, and choosing the origin of the longitudinal x-axis at the left support, the differential equation and boundary conditions become[18]

$$y'' + \frac{P}{EI}y = 0 \qquad (44a)$$

and

$$y(0) = 0; \quad y(L) = 0 \qquad (44b)$$

in which the primes indicate differentiation with respect to x.

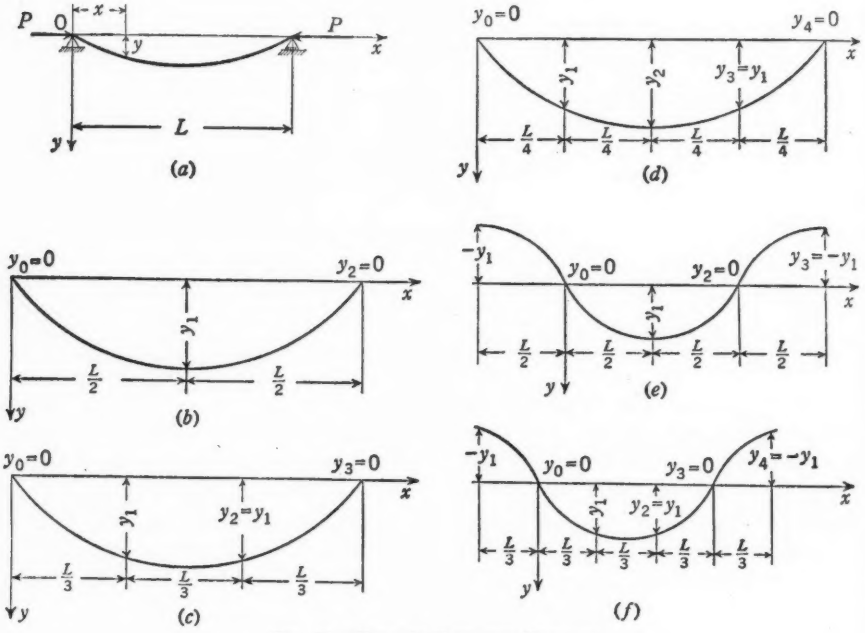

Fig. 11.—Euler's Buckling Problem

(1) *First Order Differences.*—Dividing the length L into n equal parts of length $h = L/n$ and letting the ordinates $y_i = f(x_i)$ of the deflection curve at the $n + 1$ separation points x_i—y_0, y_1, y_2, $\cdots$, y_u, y_l, y_i, y_r, y_{rr}, $\cdots$, y_{n-2}, y_{n-1}, y_n—the second derivative of y at x_i is approximated by Eq. 15*b*; thus: $y''_i \doteq \dfrac{y_l - 2\,y_i + y_r}{h^2}$.

Substitution of the latter expression in Eq. 44*a* gives the statement of the boundary problem in terms of differences:

$$y_l + b_n\,y_i + y_r = 0 \qquad (45a)$$

and

$$y_0 = 0; \qquad y_n = 0 \qquad (45b)$$

[18] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, paragraph 12.

in which

$$b_n = \frac{P_n h^2}{E I} - 2 = \frac{P_n L^2}{n^2 E I} - 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (46)$$

and P_n is the approximate value of P_{cr} obtained in the approximation with n subintervals.

Approximation $n = 2$ (Fig. 11(b)).—Eq. 45a at $x = \frac{L}{2}$ yields: $0 + b_2 y_1 + 0 = 0$. Therefore, $b_2 = 0$, from which, letting—

$$k_n = \frac{P_n}{E I/L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

—the first approximation k_2 and the corresponding error become

$$k_2 = n^2(b_n + 2)]_{n=2} = 8 \quad (-19\%).$$

Approximation $n = 3$.—Noticing that the deflection curve is symmetrical with respect to the middle point of the beam (Fig. 11(c)) and applying Eq. 45a at $x = \frac{L}{3}$: $0 + b_3 y_1 + y_1 = 0$. Therefore, $b_3 = -1$ and $k_3 = 9$ (-9%).

Approximation $n = 4$.—Eq. 45a at $x = L/4, L/2$ becomes: $0 + b_4 y_1 + y_2 = 0$ and $y_1 + b_4 y_2 + y_1 = 0$. The determinantal equation is: $\begin{vmatrix} b_4 & 1 \\ 2 & b_4 \end{vmatrix} = b^2_4 - 2 = 0$. Therefore, $b_4 = \pm\sqrt{2}$. Choosing the value of b_4 that makes k_4 smaller, $k_4 = 9.37258$ (-5%).

Approximation $n = 5$.—Eq. 45a at $x = L/5, 2L/5$ becomes: $0 + b_5 y_1 + y_2 = 0$ and $y_1 + b_5 y_2 + y_2 = 0$. The corresponding determinantal equation is: $\begin{vmatrix} b_5 & 1 \\ 1 & b_5 + 1 \end{vmatrix} = b^2_5 + b_5 - 1 = 0$. Choosing the root of this quadratic equation, which makes k_5 smaller: $b_5 = -1.62$ and $k_5 = 9.54918$ (-3.2%).

Approximation $n = 6$.—Eq. 45a at $x = L/6, L/3, L/2$ becomes: $0 + b_6 y_1 + y_2 = 0$; $y_1 + b_6 y_2 + y_3 = 0$; and $y_2 + b_6 y_3 + y_2 = 0$. The corresponding determinantal equation is: $\begin{vmatrix} b_6 & 1 & 0 \\ 1 & b_6 & 1 \\ 0 & 2 & b_6 \end{vmatrix} = b_6(b^2_6 - 3) = 0$, from which $b_6 = -1.73$ and $k_6 = 9.64616$ (-2.3%).

Approximation $n = 7$.—Eq. 45a at $x = L/7, 2L/7, 3L/7$ becomes: $0 + b_7 y_1 + y_2 = 0$; $y_1 + b_7 y_2 + y_3 = 0$; and $y_2 + b_7 y_3 + y_2 = 0$. The determinantal equation is: $\begin{vmatrix} b_7 & 1 & 0 \\ 1 & b_7 & 1 \\ 0 & 1 & b_7 + 1 \end{vmatrix} = b^3_7 + b^2_7 - 2 b_7 - 1 = 0$, from which $b_7 = -1.80$ and $k_7 = 9.70506$ (-1.7%). The correct value of K is $\pi^2 = 9.86960$.

A large number of successive approximations has been computed in the foregoing example to investigate the mode of approach of k_n toward K, but it should be noticed that, since the approximations $n = 3,5,7$ involve determinants of the same order as the approximations $n = 2,4,6$, in practice only the odd approximations would be computed.

The h^2-extrapolation formula (Eq. 41) with Table 1 and the (h^2,h^4)-extrapolation formula (Eq. 42) with Table 2 give the results in Table 5.

The extrapolated values are so near the true value that in practice two approximations and an h^2-extrapolation would give a result more than sufficiently accurate.

TABLE 5.—EXTRAPOLATIONS FOR THE CASE OF A SIMPLY SUPPORTED BEAM WITH CONSTANT MOMENT OF INERTIA

n	h^2-extrapolations	$e(\%)$	n	(h^2,h^4)-extrapolations	$e(\%)$
2,3	9.80000	−0.705	2,3,4	9.86881	−0.0080
3,4	9.85161	−0.182	3,4,5	9.86962	+0.0002
4,5	9.86314	−0.065	4,5,6	9.86932	−0.0028
5,6	9.86657	−0.031	5,6,7	9.86974	+0.0014
6,7	9.86817	−0.014	....	.....	.....

(2) *Second Order Differences.*—Approximating y'' by Eq. 16*b* the boundary value problem (Eqs. 44) in terms of second order differences, becomes

$$-y_{ll} + 16\,y_l + b_n\,y_i + 16\,y_r - y_{rr} = 0 \quad \ldots\ldots\ldots\ldots (48a)$$

and

$$y_0 = 0; \qquad y_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (48b)$$

in which

$$b_n = \frac{12\,P_n\,L^2}{n^2\,E\,I} - 30 = \frac{12\,k_n}{n^2} - 30 \quad \ldots\ldots\ldots\ldots\ldots (49)$$

Since the difference equation (Eq. 48*a*) involves the values of y two points to the left and two points to the right of x_i, the values of y_{-1} and y_{n+1} must be known before the equation can be applied at x_1 and x_{n-1}. These values are easily determined by using first order differences in expressing the boundary conditions at x_0 and at x_n. At these points the beam is simply supported and, hence, $y_0 = y_n = 0$ and $y''_0 = y''_n = 0$. Expressing y'' by first order differences (Eq. 15*b*): $h^2\,y''_0 = y_{-1} - 2\,y_0 + y_1 = y_{-1} + y_1 = 0$ and $h^2\,y''_n = y_{n-1} - 2\,y_n + y_{n+1} = y_{n-1} + y_{n+1} = 0$. Hence,

$$y_{-1} = -\,y_1; \qquad y_{n+1} = -\,y_{n-1} \quad \ldots\ldots\ldots\ldots\ldots (50)$$

These relations state the geometrical fact that x_0 and x_n are inflection points of the y-curve.

Proceeding as was done before with first order differences, the following results are obtained:

Approximation $n = 2$.—By Eq. 48*a* at $x = L/2$: $y_1 + 0 + b_2\,y_1 + 0 + y_1 = 0$; $b_2 = -2$; and $k_2 = 9.33$ (-5.47%).

Approximation $n = 3$.—By Eq. 48*a* at $x = L/3$: $y_1 + 0 + b_3\,y_1 + 16\,y_1 + 0 = 0$; $b_3 = -17$; and $k_3 = 9.75$ (-1.21%).

Approximation $n = 4$.—By Eq. 48*a* at $x = L/4, L/2$: $y_1 + 0 + b_4\,y_1 + 16\,y_2 - y_1 = 0$ and $0 + 16\,y_1 + b_4\,y_2 + 16\,y_1 + 0 = 0$, from which $\begin{vmatrix} b_4 & 16 \\ 32 & b_4 \end{vmatrix} = b^4 - 2 \times 16^2 = 0$ and $b_4 = -22.63$. Therefore, $k_4 = 9.83$ (-0.40%).

Extrapolations.—Using the h^4-extrapolation formulas and the (h^4,h^6)-extrapolation formulas (Eqs. 43 and Tables 3 and 4), the following extrapolated values are obtained: $k_{2,3} = 9.85(-\ 0.15\%)$; $k_{3,4} = 9.87(-\ 0.02\%)$; and $k_{2,3,4} = 9.8695(-\ 0.0009\%)$.

The evaluation of the second order approximations and extrapolations does not require an increased amount of labor, but the second order approximations are much nearer to the true value (π^2) of K than are the first order approximations. In problems of this type it may well pay to use second order differences directly.

B. Simply Supported Beam with Variable Moment of Inertia

In the previous problem, if the moment of inertia I varies according to the law (in which I_o is a constant),

$$I(x)\begin{cases} = I_o\left(1+\dfrac{2x}{L}\right) & 0 \leqq x \leqq \dfrac{L}{2} \\ = I_o\left(3-\dfrac{2x}{L}\right) & \dfrac{L}{2} \leqq x \leqq L \end{cases}$$

Eq. 44*a* becomes, by Eq. 15*b*,

$$y_l + \left(\frac{a_n}{1+2x/L} - 2\right) y_i + y_r = 0 \left(0 \leqq x \leqq \frac{L}{2}\right) \ldots\ldots\ldots (51)$$

in which

$$a_n = \frac{P_n L^2}{n^2 E I_o} = \frac{k_n}{n^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

—whereas the boundary conditions remain $y_0 = 0$ and $y_n = 0$. The successive approximations are obtained as before, remembering that y is symmetrical about the middle point of the beam. Table 6 gives the results of the computations.

TABLE 6.—Extrapolations for the Case of a Simply Supported Beam with Variable Moment of Inertia

n	k_n	$e_n(\%)$	h^2-Extrapolations			(h^2,h^4)-Extrapolations		
			n	k	$e(\%)$	n	k	$e(\%)$
2	16.0000	−3.23	2,4	16.0000	−3.23	2,4,6	16.5114	−0.13
3	15.0000	−9.27	3,5	16.3933	−0.85	3,5,7	16.4964	−0.22
4	16.0000	−3.23	4,6	16.4546	−0.48	....	....	....
5	15.8917	−3.83	5,7	16.4774	−0.34	....	....	....
6	16.2526	−1.70	....	....	....	....	....	....
7	16.1786	−2.15	....	....	....	....	....	....

The approach of k_n to K is one sided, but not monotonic, whereas the separate sequences of even and odd k_n are both monotonic. The extrapolations of Table 6 are obtained by these sequences.

The energy method applied to this problem with an assumed sinusoidal deflection gives an upper bound $k = 16.5342$. Hence it is probable that the average of the two (h^2,h^4)-extrapolations is no more than 0.2% off.

C. Lateral Buckling of a Simply Supported Beam of Narrow Cross Section

A beam of narrow rectangular cross section is simply supported at $x = \pm L/2$, while the rotation of its ends around the axis of the beam is prevented. The beam buckles laterally under the action of a vertical load P at $x = 0$. If B is the smaller of the two flexural rigidities of the beam in its principal planes, C is the torsional rigidity, and β is the rotation of a section x, the boundary problem becomes[19]

$$\beta'' + \frac{P^2 L^2}{4 B C}\left(\frac{1}{2} - \frac{x}{L}\right)^2 \beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (53a)$$

and

$$\beta\left(\frac{L}{2}\right) = \beta\left(-\frac{L}{2}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (53b)$$

The corresponding difference problem becomes, by Eq. 15*b*,

$$\beta_l + \left[a_n\left(\frac{1}{2} - \frac{x}{L}\right)^2 - 2\right]\beta_i + \beta_r = 0 \quad \ldots\ldots\ldots\ldots (54a)$$

and

$$\beta_0 = 0; \quad \beta_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (54b)$$

in which

$$a_n = \frac{P^2{}_n L^4}{4 n^2 B C} = \frac{k^2{}_n}{(2n)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

Table 7 gives the results of the computations:

TABLE 7.—Extrapolations for the Case of Lateral Buckling of a Narrow Simply Supported Beam

n	k_n	e_n(%)	h^2-Extrapolations			(h^2,h^4)-Extrapolations		
			n	k	e (%)	n	k	e (%)
2	11.3137	−33.2	2,4	16.21	−4.3	2,4,6	16.905	−0.09
3	18.0000	+ 6.32	3,5	16.62	−1.8	3,5,7	16.933	+0.02
4	14.9827	−11.82	4,6	16.83	−0.6	....	....	....
5	17.1190	+ 1.11	5,7	16.88	−0.3	....	....	....
6	16.0075	− 5.40	....	....	....	....	....	....
7	17.0000	+ 0.41	....	....	....	....	....	....

The true value of K is 16.93. In this particular problem the approach of k_n to K is of the oscillatory type, the even k_n being smaller, the odd k_n being larger than K, and the extrapolations have been obtained by the even and odd sequences.

[19] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 250.

D. Lateral Buckling of a Cantilever Beam of Narrow Cross Section

If the beam of the previous example is built in at $x = 0$ and loaded by a transverse load P at $x = L$, the corresponding boundary value problem becomes[20]

$$\beta'' + \frac{P^2 L^2}{B C}\left(1 - \frac{x}{L}\right)^2 \beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (56a)$$

and

$$\beta(0) = 0; \qquad \beta'(L) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (56b)$$

and, by Eq. 15*b*, the finite difference problem reduces to

$$\beta_l + \left[a_n \left(1 - \frac{x}{L}\right)^2 - 2\right] \beta_i + \beta_r = 0 \quad \ldots\ldots\ldots\ldots (57a)$$

and

$$\beta_0 = 0; \qquad \beta_{n+1} = \beta_{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (57b)$$

in which

$$a_n = \frac{P^2{}_n L^4}{n^2 B C} = \frac{k^2{}_n}{n^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (58)$$

Table 8 gives the results of the computations.

TABLE 8.—Extrapolations for the Case of Lateral Buckling of a Narrow Cantilever Beam

n	k_n	e_n (%)	h^2-Extrapolations			(h^2,h^4)-Extrapolations		
			n	k	e (%)	n	k	e (%)
2	4.000	−3.2	3,4	3.992	−0.52	2,3,4	4.030	+0.42
3	3.933	−1.9	4,5	4.006	−0.17	3,4,5	4.014	+0.025
4	3.959	−1.3	....	....	....	....	....	....
5	3.976	−0.92	....	....	....	....	....	....

The correct value of K is 4.013 and the convergence of k_n to K is monotonic after $n = 3$. The error is of the type $e = -a h^2 + b h^4$. The h^2-extrapolation may be applied for $n = 3,4$ and $n = 4,5$, whereas the (h^2,h^4)-extrapolation may be applied to any sequence, and in particular to $n = 2,3,4$ and $n = 3,4,5$.

E. Buckling of a Simply Supported Square Plate Under Uniform Compression

A square plate with sides L and flexural rigidity D, simply supported along its boundary, buckles under a uniform compression N per unit of boundary length. Taking the x,y-axes parallel to the plate sides with origin at one of its corners, and calling $w(x,y)$ the deflection of the plate, the corresponding boundary value problem becomes[21]

$$\nabla^4 w + \frac{N}{D} \nabla^2 w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (59a)$$

[20] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 245.

[21] *Ibid.*, p. 333.

and, on the boundary,

$$\nabla^2 w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (59b)$$

or, letting $\nabla^2 w = z$,

$$\nabla^2 z + \frac{N}{D} z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60a)$$

and, on the boundary,

$$z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60b)$$

First Order Differences.—Making use of first order differences (Eq. 19*c* and Fig. 4(*c*)) the corresponding difference problem becomes

$$z_r + z_l + z_a + z_b + c_n z_i = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (61a)$$

and, on the boundary,

$$z_i = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (61b)$$

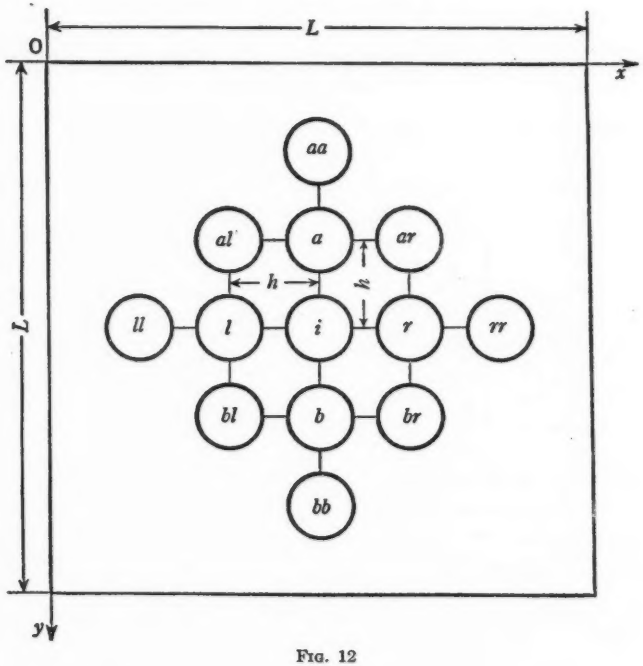

FIG. 12

in which the subscripts indicate the grid points according to Fig. 12, and

$$c_n = \frac{N_n L^2}{n^2 D} - 4 = \frac{k_n}{n^2} - 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (62)$$

Approximation $n = 2$.—Applying Eq. 61*a* at $x,y = L/2$: $c_2 = 0$ and $k_2 = 16.00$ ($-$ 19%).

Approximation $n = 3$.—Applying Eq. 61*a* at $x,y = L/3$: $z_1 + 0 + z_1 + 0 + c_3 z_1 = 0$ and $c_3 = -2$ and $k_3 = 18.00$ ($-$ 8.8%).

Approximation $n = 4$.—Applying Eq. 61a at $x,y = L/2, L/4$: $\begin{vmatrix} c_4 & 2 & 0 \\ 2 & c_4 & 1 \\ 0 & 4 & c_4 \end{vmatrix} = 0.$

Therefore, $c_4 = -2.8284$ and $k_4 = 18.75$ (-5.1%). By extrapolation: $k_{2,3} = 19.60$ (-0.71%); $k_{3,4} = 19.71$ (-0.15%); and $k_{2,3,4} = 19.75$ $(+0.051\%)$. The true value of K is 19.739.

F. Buckling of a Square Plate with Clamped Edges Under Uniform Compression

If the square plate of Example E has clamped edges, the boundary value problem becomes[22]

$$\nabla^4 w + \frac{N}{D} \nabla^2 w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (63a)$$

and, on the boundary,

$$w = 0; \qquad \frac{\partial w}{\partial \nu} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (63b)$$

ν being the normal to the boundary. By the first order difference operators of Figs. 4(b) and 4(d), the corresponding difference problem becomes

$$w_{rr} + w_{ll} + w_{aa} + w_{bb} + 2\,(w_{br} + w_{bl} + w_{ar} + w_{al}) + c_n\,(w_r + w_l + w_a + w_b) - (4\,c_n + 12)\,w_i = 0 \quad \ldots (64a)$$

and

$$\left.\begin{aligned} w_i = 0, \quad w_l = w_r \qquad & \text{on } x = 0, \quad x = L \\ w_i = 0, \quad w_a = w_b \qquad & \text{on } y = 0, \quad y = L \end{aligned}\right\} \quad \ldots\ldots\ldots (64b)$$

in which

$$c_n = \frac{N_n L^2}{n^2 D} - 8 = \frac{k_n}{n^2} - 8 \quad \ldots\ldots\ldots\ldots\ldots\ldots (65)$$

Table 9 gives the results of the computations.

TABLE 9.—Extrapolations for the Buckling of a Square Plate with Clamped Edges

n	k_n	e_n (%)	h^2-Extrapolations			(h^2,h^4)-Extrapolations		
			n	k	e (%)	n	k	e (%)
2	24.000	−56.0	2,3	45.600	−12.97	2,3,4	52.129	−0.49
3	35.000	−31.0	3,4	50.497	− 3.61	3,4,5	52.156	−0.44
4	43.342	−17.4	4,5	51.558	− 1.58	4,5,6	53.243	+1.64
5	45.660	−12.8	5,6	52.494	− 0.21	2,4,6	52.525	+0.21
6	47.748	− 9.0	....	....	....	....	....	....

The literature on the buckling of rectangular built-in plates is extensive, but no rigorous solution of the problem has been given. For the case of the present example, A. Weinstein[4] gives very close upper and lower bounds whose average, 52.385, has been assumed as a true value for the evaluation of the

[22] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, pp. 305, 298, and 364.

errors in the extrapolations. (The average of the bounds given by Mr. Timoshenko[23] is 52.455 and differs by 0.13% from the average of Mr. Weinstein's closer bounds.) It is interesting to notice that all the h^2-extrapolations are from below, whereas some (h^2,h^4)-extrapolations overshoot the mark. The computation of the higher approximations is very cumbersome, but the (h^2,h^4)-extrapolation gives excellent results by the first three approximations.

G. Buckling of a Rectangular Plate with Clamped Edges Under Uniform Compression

The problem of Example F can easily be extended to the case of a rectangular plate with sides of lengths $2\,L$ and L parallel to x and y, respectively. By Eq. 19*b* for $r = 1/2$ and its square, the following equation for the difference problem is obtained:

$$w_{rr} + w_{ll} + 16\,(w_{bb} + w_{aa}) + 8\,(w_{br} + w_{bl} + w_{ar} + w_{al}) + c_n\,(w_r + w_l) + 4\,c_n\,(w_b + w_a) - (10\,c_n + 66)\,w_i = 0 \quad (66)$$

with

$$w_i = 0, \quad w_l = w_r \qquad \text{on } x = 0, \quad x = 2\,L \quad (67a)$$

and

$$w_i = 0, \quad w_a = w_b \qquad \text{on } y = 0, \quad y = L \quad (67b)$$

in which

$$c_n = \frac{4\,N_n\,L^2}{n^2\,D} - 20 = \frac{4\,k_n}{n^2} - 20 \quad (68)$$

TABLE 10.—Extrapolations for the Buckling of a Rectangular Plate with Clamped Edges

n	$4\,k_n$	h^2-Extrapolations			(h^2,h^4)-Extrapolations		
		n	$4\,k_n$	e (%)	n	$4\,k_n$	e (%)
2	67.200	2,3	137.400	−11.5	2,3,4	153.906	−0.84
3	106.200	3,4	149.779	− 3.53	3,4,5	154.768	−0.32
4	125.266	4,5	152.972	− 1.47	....	....	....
5	135.240	....	....	....	....	....	....

The results of the computations are presented in Table 10.[24] The errors have been computed assuming as true value the buckling load obtained by the energy method with a deflection function—$w = \frac{w_0}{4}\left(1 - \cos\frac{2\,\pi\,x}{\rho\,L}\right)\left(1 - \cos\frac{2\,\pi\,y}{L}\right)$—in which ρ is the ratio of the side parallel to x to the side parallel to y. The expression for this upper bound of the load is $\frac{4\,N\,L^2}{\pi^2\,D} = \frac{16}{3}\frac{3\,\rho^4 + 2\,\rho^2 + 3}{\rho^4 + \rho^2}$ which, for $\rho = 2$, gives $4\,K = 155.269$.

[23] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 365.

[24] "Buckling Loads for Built-In Rectangular Plates by Finite Differences," by J. Gold, thesis submitted to Columbia Univ., at New York, N. Y., in 1947, in partial fulfilment of the requirements for the degree of Master of Science.

H. Buckling of a Simply Supported Skew Plate Under Uniform Compression

A simply supported 45° skew plate, with sides L parallel to x and 45° sides $\sqrt{2}\,L$, buckles under a uniform compression N per unit of boundary length. The corresponding boundary value problem is identical with the problem of Example G.

A 45° skew plate may be covered by both a rectangular and a skew grid. Using a skew grid with $r = \sqrt{2}$ and $\alpha = 45°$, Eq. 23, or Fig. 7, Section 4, yields the difference problem:

$$2\,z_r + 2\,z_l + 4\,z_a + 4\,z_b - z_{ar} - z_{bl} + z_{br} + z_{al} + c_n\,z_i = 0 \quad \ldots (69a)$$

and, on the boundary,

$$z_i = 0 \quad \ldots (69b)$$

in which

$$c_n = \frac{2}{n^2}\frac{N_n\,L^2}{D} - 12 = \frac{2}{n^2}\,k_n - 12 \quad \ldots (70)$$

Table 11(a) gives the results of the computations. The successive values of

TABLE 11.—Extrapolations for the Case of the Buckling of a 45° Skew Plate

n	(*a*) Skew Coordinates					(*b*) Square Grid				
	k_n	h^2-Extrapolations		(h^2,h^4)-Extrapolations		k_n	h^2-Extrapolations		(h^2,h^4)-Extrapolations	
		n	k	n	k		n	k	n	k
2	24	2,3	28.73	2,3,4	27.51	16.00	2,3	25.76	2,3,4	26.68
3	26.63	3,4	27.82	3,4,5	27.21	21.42	3,4	26.45	3,4,5	26.72
4	27.15	4,5	27.43	....	....	23.62	4,5	26.62	....	....
5	27.25	....	....	....	....	24.70	....	....	....	....

k_n obtained by a square grid are given in Table 11(b), together with the h^2-extrapolations and the (h^2,h^4)-extrapolations.

No rigorous solution of this problem is known and a value of $k = 27$ may be accepted as plausible.

It must be noticed that, to obtain the $n = 5$ approximation in Cartesian coordinates, it is necessary to solve a determinantal equation of the eighth order. Skew coordinates are much better adapted to the problem and lead to the final result with a considerably smaller amount of labor.

I. Buckling of a Simply Supported, Equilateral, Triangular Plate Under Uniform Compression

A simply supported, equilateral, triangular plate, of sides L, buckles under a uniform compression N per unit of boundary length. The boundary value problem to be solved is identical with the problem of Example E, but in this case the plate may be covered by a triangular grid. By Eq. 27 or Fig. 9(a),

the difference problem becomes

$$\Sigma z_e + c_n z_i = 0 \quad \ldots\ldots (71a)$$

and, on the boundary,

$$z_i = 0 \quad \ldots\ldots (71b)$$

in which

$$c_n = \frac{N L^2}{n^2 D} - 6 = \frac{3}{2 n^2} k_n - 6 \quad \ldots\ldots (72)$$

and Σz_e is the sum of the values of z at the vertices of the hexagon surrounding the ith point.

TABLE 12.—EXTRAPOLATIONS FOR THE CASE OF THE BUCKLING OF A SIMPLY SUPPORTED, EQUILATERAL, TRIANGULAR PLATE UNDER UNIFORM COMPRESSION

n	k_n	e_n (%)	h^2-EXTRAPOLATIONS			(h^3,h^4)-EXTRAPOLATIONS		
			n	k	e (%)	n	k	e (%)
3	36.00	−31.6	3,4	51.25	−2.64	3,4,5	52.56	−0.15
4	42.67	−19.0	4,5	52.09	−1.04	4,5,6	52.67	+0.06
5	46.06	−13.7	5,6	52.41	−0.44	5,6,7	52.641	+0.002
6	48.00	− 8.8	6,7	52.52	−0.23	....	....	....
7	49.20	− 6.5	....	....	....	....	....	....

Table 12 gives the results of the computations. The correct value of K is 52.64.[25] Making use of finite difference operators in irregular triangular coordinates[15] a triangular plate of any shape may be similarly solved.

J. BUCKLING OF A SIMPLY SUPPORTED HEXAGONAL PLATE UNDER UNIFORM COMPRESSION

The buckling load of a regular hexagonal plate, simply supported and subject to uniform compression, may be computed by Eqs. 71 and 72 of Example I.

TABLE 13.—EXTRAPOLATIONS FOR THE CASE OF THE BUCKLING OF A SIMPLY SUPPORTED HEXAGONAL PLATE UNDER UNIFORM COMPRESSION

n	m	k_n	h^2-EXTRAPOLATIONS		(h^2,h^4)-EXTRAPOLATIONS	
			n	k	n	k
1	1	4	1,2	7.04	1,2,3	7.18
2	2	6.28	2,3	7.16	2,3,4	7.25
3	4	6.77	3,4	7.23	....	....
4	6	6.97	....	....	....	....

The values of the successive approximations are given in Table 13, in which m is the number of body points considered and hence represents the order of the determinant to be solved; 7.20 may be taken as a plausible value for k.

[25] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 371.

K. Symmetrical Buckling of an Infinitely Long Cylindrical Shell Under Uniform Axial Compression

An infinitely long, thin, cylindrical shell of radius R, thickness h, flexural rigidity D, and Young's modulus E, buckles under a uniform axial compression N per unit length. Assuming that the shell buckles into half waves of length L, the corresponding boundary value problem is[26]

$$y'''' + \frac{N}{D} y'' + \frac{E h}{D R^2} y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (73a)$$

and

$$y(0) = y(L) = 0; \qquad y''(0) = y''(L) = 0 \quad \ldots\ldots\ldots\ldots (73b)$$

in which y is the radial displacement.

By Eqs. 15, the corresponding difference problem becomes

$$y_{ii} + (a_n - 4)\, y_i + (6 - 2\, a_n + b_n)\, y_i + (a_n - 4)\, y_r + y_{rr} = 0 \quad \ldots (74a)$$

and

$$y_{-1} = -\, y_1; \qquad y_{n+1} = -\, y_{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (74b)$$

in which

$$a_n = \frac{N_n L^2}{n^2 D}; \qquad b_n = \frac{L^4}{n^4} \frac{E h}{D R^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (75)$$

Approximation $n = 2$.—Applying Eq. 73a at $x = L/2$: $4 - 2\, a_2 + b_2 = 0$ or $N_2 = D \left(\frac{8}{L^2} - \frac{E h}{D R^2} \frac{L^2}{8} \right)$.

Since the shell is infinitely long, it will buckle into an infinite number of half waves, whose length L is such as to make N a minimum. Letting

$$\frac{8}{L^2} = z; \qquad \frac{E h}{D R^2} = c \quad \ldots\ldots\ldots\ldots\ldots\ldots (76)$$

the function, $N_2 = D \left(z + \frac{c}{z} \right)$, will be minimum with respect to L, and hence with respect to z, when $\frac{dN_2}{dz} = D \left(1 - \frac{c}{z^2} \right) = 0$. Therefore,

$$z = \sqrt{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (77a)$$

Thus

$$(N_2)_{\min} = 2\, D \sqrt{c} = \frac{2}{R} \sqrt{D E h} \quad \ldots\ldots\ldots\ldots\ldots\ldots (77b)$$

This value of N_{cr} is rigorously correct and is obtained in all the successive approximations as the value of $(N_n)_{\min}$.

[26] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc. New York, N. Y., 1st Ed., 1936, p. 441.

By Eqs. 76 and 77*a*, the half-wave length L_2 corresponding to $(N_2)_{min}$ is: $L_2 = 8/\sqrt{c}$; and, letting

$$k_n = L_n \Big/ \sqrt[4]{\frac{D R^2}{E h}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (78)$$

the corresponding value of k_2 is: $k_2 = 2.8284$ and $e_2 = -9.96\%$.

Approximations $n = 3$, 4, and 5.—For $n = 3$, applying Eq. 74*a* at $x = L/3$: $1 - a_3 + b_3 = 0$ or $N_3 = D(9/L^2 + c\,L^2/9)$. The half-wave length corresponding to N_3 gives: $k_3 = 3$ and $e_3 = -4.51\%$. Similarly, for $n = 4$ and $n = 5$: $k_4 = 3.06$, $e_4 = -2.6\%$; $k_5 = 3.09$; and $e_5 = -1.64\%$. By extrapolation: $k_{2,3} = 3.1373$ (-0.14%); $k_{3,4} = 3.1371$ (-0.14%); $k_{4,5} = 3.1433$ $(+0.055\%)$; $k_{2,3,4} = 3.1370$ (-0.15%); and $k_{3,4,5} = 3.1468$ $(+0.17\%)$. The correct value of k is π.

11. CONCLUSIONS

In this paper (which is of an essentially experimental character) the method of finite differences is applied mostly to problems with a known solution to check the rapidity and the type of convergence, as well as the validity of the suggested extrapolation procedures. However, the scope of the method is such as to allow the attack of a large variety of buckling problems. Among the advantages of the method are the following:

(*a*) Problems with different types of boundary conditions can be solved by the same basic procedures with a comparable amount of labor in all cases.

(*b*) The variability of the moment of inertia of beams and plates does not require special procedures and in general does not increase materially the amount of labor involved.

(*c*) Beams with compressive loads P variable along the axis of the beam and plates with known variable distribution of stress in the plane of the plate may be considered without increased difficulty.

(*d*) Anisotropic plates fall under the scope of the method.

(*e*) Plates of irregular shapes may be dealt with by a suitable set of coordinates, using the equations that best fit the problem in each case.

(*f*) The method does not require a knowledge of higher mathematics. It is interesting to notice that, with the exception of relaxation procedures, all other methods of solution, both approximate and rigorous, require instead the use of integral calculus, differential equations, or series expansion.

(*g*) The method is applicable to all characteristic value problems involving linear differential equations, ordinary or partial, with constant or variable coefficients.

There are also limitations and difficulties inherent in the method:

(1) The method is not applicable when the distribution of longitudinal stress in the beam or of plane stress in the plate is unknown. This limitation is common to all other methods, except the energy method.

(2) There is no way of knowing whether the extrapolated values are greater or smaller than the true value.

(3) The literal evaluation of the determinantal equation for the higher approximations may be cumbersome.

This difficulty may be avoided as explained in Section 8, but it is interesting to notice that all series solutions, including those used in connection with the energy method, lead to determinantal equations and hence involve the same amount of labor in the final step of the computations, although sometimes a larger amount of labor is required in the derivation of the determinant itself. Moreover, most of the other methods do not lend themselves easily to extrapolation procedures. (Relaxation procedures, also involving finite differences, seem to be more cumbersome than the present method, since they require the simultaneous evaluation of the deflection function, which seldom needs to be known in practice. The third book by Mr. Southwell on relaxation methods, which is planned to cover characteristic value problems, has not been published up to the time of this writing (1949) and it is therefore impossible to make a final comparison between the two methods.)

The width of its scope and the simplicity of its application should recommend the finite difference method to the attention of the engineer and of the applied mathematician.

12. ACKNOWLEDGMENTS

The writer is highly endebted to J. L. Bogdanoff, instructor in civil engineering, Columbia University at New York, N. Y., for calling his attention to Mr. Richardson's paper,[10] to his colleagues in the Department of Civil Engineering at Columbia University, and particularly to R. D. Mindlin, Assoc. M. ASCE, professor of civil engineering, Columbia University, for encouragement and suggestions in the writing of this paper. The values of the buckling loads for Example G of Section 10 are taken from a thesis by J. Gold, Jun. ASCE, entitled "Buckling Loads for Built-in Rectangular Plates by Finite Differences." The value of k_n (= 27.25) for $n = 5$ in Table 11(a) was computed by A. J. Del Mastro, Jun. ASCE.

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PAPERS

A STUDY OF END CONNECTIONS FOR STRUTS

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Synopsis

Results of tests to evaluate the effect of different types of end connections upon the static and fatigue strengths of struts are described in this paper. Tubes, single angles, and double angles were tested with the usual types of end connections employed for these members as well as some types not in common use. Strain measurements were made to determine the distribution of stress in the members and their connections.

It was found that the rigidity of the connection, degree of eccentricity, and the amount of stress concentration caused by changes of section at the column ends had a marked influence on the static and fatigue strengths.

Although the tests were made on aluminum alloy members, it is believed that the comparisons drawn apply to members of other materials also.

Introduction

One of the difficult problems in the design of latticed structures concerns the end connections of the web members. In lightweight structures such as towers, booms, and relatively light roof trusses, structural angles are generally used, and the members are connected by bolts or rivets through one leg. It is generally recognized that this method of attachment produces an eccentricity of loading so that: (1) The deformations are larger than estimated on the basis of uniform stress in the member; (2) the maximum stress is somewhat larger than the average stress; and (3) the ultimate strength is less than that of a similar member with axial load.[3,4] The performance of the structure is influenced by the performance of the individual members, so it is reasonable

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by May 1, 1950.

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[3] "Model Tests of Latticed Structural Frames," by E. C. Hartmann, R. L. Moore, and Marshall Holt, *Technical Paper No. 2*, Aluminum Research Laboratories, New Kensington, Pa., 1938.

[4] "Tests of 28-Foot Span Aluminum Alloy Trusses," by R. L. Templin, E. C. Hartmann, and H. N Hill, *Technical Paper No. 3*, Aluminum Research Laboratories, New Kensington, Pa., 1940.

to believe that the analysis of the structure will be simpler and more satisfactory if the behavior of the individual members is more thoroughly understood. In addition, the performance of the structure might be improved if the eccentricity of loading of the individual members is eliminated or at least decreased.

The round tube with symmetrical connections appears to be a desirable type of member, but the difficulties of making suitable end connections have discouraged the use of this type of member in riveted construction. The use of tubular members in welded construction, however, has been discussed and advocated.[5,6]

An experimental investigation involving both static and repeated loadings was undertaken at the Aluminum Research Laboratories in New Kensington, Pa., to study the relative merits of various types of end connections for round-tubular struts, single-angle struts, and double-angle struts. Although aluminum alloy members were used in the tests, it is believed that the comparisons drawn apply to other metals also.

Description of Specimens

The specimens used in this investigation are listed in Tables 1 and 2. Types A, B, C, and D were $2\frac{1}{4}$ in. in outside diameter, with walls 0.134 in. thick, and were seamless, drawn round tubes of aluminum alloy 52S in the H32 or stabilized quarter-hard temper. Types E, F, and G were single $3\frac{1}{2}$-in. by $3\frac{1}{2}$-in. by $\frac{1}{4}$-in. structural angles of extruded aluminum alloy 17S in the T4 or fully heat treated and naturally aged temper. (Abbreviations such as 17S and 52S denote the trade designations by which the alloy is commonly recognized.[7]) Type H was composed of two extruded 17S-T4 unequal-leg angles $2\frac{1}{2}$ in. by 2 in. by $\frac{3}{16}$ in. separated by $\frac{1}{4}$-in.-thick washers. The specimens were loaded through a gusset plate connection.

The tubes of types A and B had flattened ends to facilitate making a riveted or bolted connection. In type A, the ends were flattened against a tube having a wall thickness of 0.138 in. and a length of 10 in. which was telescoped into the end of the member to serve as a filler. The flattening was done cold with two blows by a steam hammer. The resulting thickness of the flattened end with the filler tube left in was about $\frac{1}{2}$ in. For the static tests, the specimens were attached, by turned studs, directly to a steel loading fixture. In type B, there was no filler tube, but a filler plate was inserted temporarily and the end of the specimen was flattened against this plate. The paddle-shaped end was split along the edges, and a gusset plate fitted snugly into the slot thus formed.

The tubes of types C and D had slotted ends but were not flattened. In type C, the ends of the tubes were fitted with segmental plugs to prevent flattening against the gusset plate when the bolts were tightened. In type D, the tube and gussets were joined by welding on both sides of the gusset plates.

[5] "Tubing Saves Steel in Long Trusses," *Engineering News-Record*, July 2, 1943, p. 37.

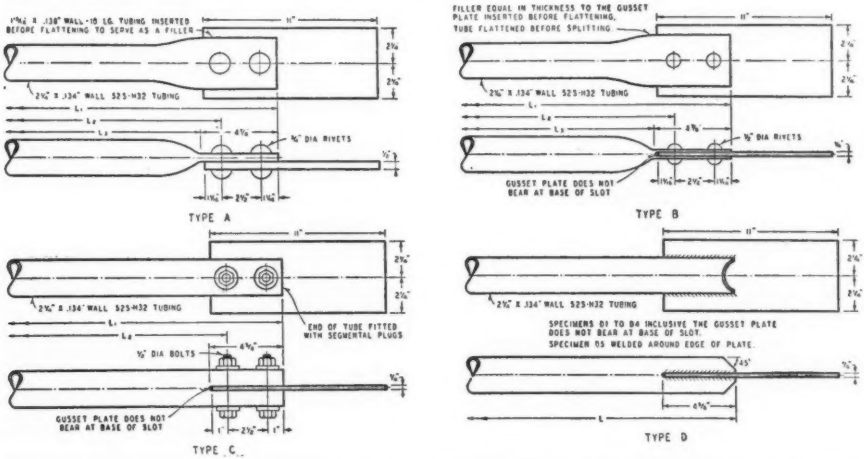
[6] "The Use of Tubular Steel Sections in Structures," by A. Quisling and C. M. Cosman (rept. prepared under the auspices of the Literature Div. of the Eng. Foundation Welding Research Committee), Supplement to *Journal*, Am. Welding Soc., March, 1943.

[7] *Transactions*, ASCE, Vol. 102, 1937, p. 1279, Table 7.

The welds were made in two passes with an oxyhydrogen flame. In specimen D5, which was tested under repeated load, the weld was continuous around the end of the gusset plate whereas in all the other specimens of type D the weld was shorter than the overlap of the tube and gusset plate.

The specimens of type E, Table 2, were simply equal-leg structural angles with the connections through only one leg, the conventional type of single-angle connection.

TABLE 1.—TUBULAR STRUTS LOADED THROUGH GUSSET PLATES


TYPE A
TYPE B
TYPE C
TYPE D
GUSSET PLATE DOES NOT BEAR AT BASE OF SLOT
END OF TUBE FITTED WITH SEGMENTAL PLUGS
SPECIMENS D1 TO D4 INCLUSIVE THE GUSSET PLATE DOES NOT BEAR AT BASE OF SLOT.
SPECIMEN D5 WELDED AROUND EDGE OF PLATE.

Specimen	Type A[a]			Type B[a]			Type C[a]		Type D[a]
	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L
1[b]	78	70⅞	68¼	68	60⅞	58¼	68	61	68
2[c]	78	70⅞	68¼	68	60⅞	58¼	68	61	68
3[c]	38⅝	31½	28⅞	28⅝	21½	18⅞	28⅝	21⅝	28⅝
4[d]	61	53⅝	51½	61	53⅝	51½	61	54	61
5[d]	....	....	....	....	....	....	....	....	61

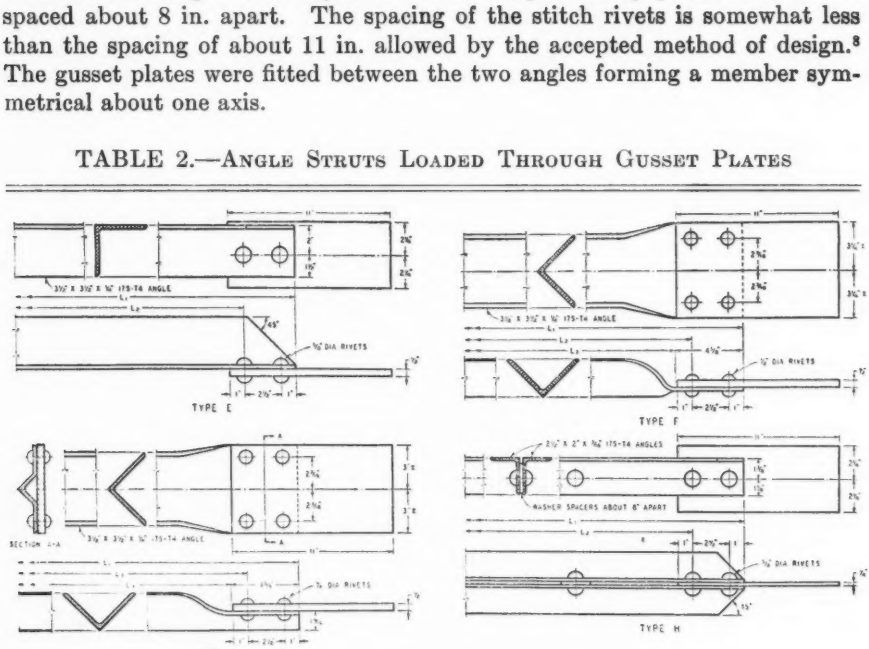
[a] Lengths, in inches; see diagrams. [b] Tested as columns with round ends. [c] Tested as columns with flat ends. [d] Fatigue tested.

The specimens of type F were equal-leg angles with flattened ends. To prevent cracking along the fillet during the flattening operation, the heel of the angle was rounded so that the material was of nearly constant thickness for a length of about 8 in. The flattening was done with a hammer while the metal was at a temperature of about 700° F. The specimens were then reheat-treated and straightened on a "gag" straightener.

The ends of the equal-leg angle specimens of type G were only partly flattened. This operation was done with the metal at room temperature using special dies in a hydraulic press.

The specimens of type H were double-angle members. The long legs of the individual angles were adjacent and were separated by $\frac{1}{4}$-in.-thick washers spaced about 8 in. apart. The spacing of the stitch rivets is somewhat less than the spacing of about 11 in. allowed by the accepted method of design.[8] The gusset plates were fitted between the two angles forming a member symmetrical about one axis.

TABLE 2.—ANGLE STRUTS LOADED THROUGH GUSSET PLATES



Specimen	Type E[a]		Type F[a]			Type G[a]			Type H[a]	
	L_1	L_2	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2
1[b]	68	61	68	61	$58\frac{3}{4}$	68	61	$58\frac{3}{4}$	68	61
2[c]	68	61	68	61	$58\frac{3}{4}$	68	61	$58\frac{3}{4}$	68	61
3[c]	$28\frac{5}{8}$	$21\frac{5}{8}$	$28\frac{5}{8}$	$21\frac{5}{8}$	$19\frac{3}{8}$	$28\frac{5}{8}$	$21\frac{5}{8}$	$19\frac{3}{8}$	78	71
4	78[c]	71[c]	61[d]	54[d]	$51\frac{3}{4}$[d]	61[d]	54[d]	$51\frac{3}{4}$[d]	61[d]	54[d]
5	$38\frac{1}{8}$[c]	$31\frac{1}{8}$[c]	61[d]	54[d]	$51\frac{3}{4}$[d]	61[d]	54[d]	$51\frac{3}{4}$[d]	61[d]	54[d]
6[d]	61	54	....	....	....	....	....	....	....	....
7[d]	61	54	....	....	....	....	....	....	....	....

[a] Lengths, in inches; see diagrams. [b] Tested as columns with round ends. [c] Tested as columns with flat ends. [d] Fatigue tested.

The elements of the cross sections are given in Table 3(*a*). The particular sections used were chosen so that the least radii of gyration of the tubes, single angles, and double angles would be approximately equal. The mechanical properties of the materials are given in Table 3(*b*). Both the tensile and compressive tests on the tubing were made on specimens of the full cross section, and the strains were measured with type F Huggenberger tensometers (gage length of 0.5 in. and mechanical multiplication of about 2,000). The tensile tests on the angles were made on standard sheet metal specimens 0.5 in. wide,[9] and

[8] "Alcoa Structural Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1948, p. 72.

[9] "Standard Methods of Tension Testing of Metallic Materials" (E8–46), *A.S.T.M. Standards*, Philadelphia, Pa., 1946, Pt. 1B, p. 297, Fig. 2.

the strains were measured with an Ewing extensometer having a gage length of 2 in. The compression tests were made on specimens of the full cross section having a slenderness ratio of about 10. The compressive yield strengths were determined from stress-strain relations obtained by measuring the relative movement of the platens of the testing machine by dial gages at the corners of the platens. This procedure has been found satisfactory by comparisons of such values of yield strength with values determined from precise stress-strain curves. The values given in Table 3(*b*) meet the requirements of federal specification QQ–A–351 for 17S shapes or the United States Navy Department specification 44T32 for 52S tubing.

TABLE 3.—STUDY OF END CONNECTIONS FOR STRUTS

Shape	Size (in.)	(*a*) Elements of Sections				(*b*) Mechanical Properties of Materials				
		Actual thickness (in.)	Area (Sq In.)		Nominal radius of gyration (in.)	Alloy and temper[b]	Strength (Lb per Sq In.)			Elongation in 2 in.(%)
			Nominal	Actual[a]			Compressive yield[c]	Tensile	Tensile yield[c]	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Round tube...	2½ × 0.134[d]	0.136	0.891	0.904	0.750	52S–H32	24,200	32,700	23,500	17.3
Angles—										
Equal leg...	3½ × 3½ × ¼	0.252	1.691	1.714	0.686[e]	17S–T4	35,800	56,700	39,100	29.5
Unequal leg.	2½ × 2 × 5/16[f]	0.198	1.630	1.726	0.871[g]	17S–T4	39,000	61,400	42,250	21.0

[a] Computed from the length and weight of representative samples and the nominal specific gravity of the material. [b] Abbreviations in Col. 7 denote the trade designation by which the alloy is commonly recognized. [c] 0.2% offset. [d] Outside diameter and wall thickness, respectively. [e] About the axis of least stiffness. [f] Two angles. [g] About the axis parallel to the long legs; angles are separated ¼ in.

The nominal composition of alloy 52S includes 2.5% magnesium, 0.25% chromium, and the remainder aluminum. The various tempers of this material are obtained by introducing certain amounts of cold work after an annealing treatment and then stabilizing by aging for 4 hours at 320° F or for 1.5 hours at 350° F. The nominal composition of alloy 17S includes 4% of copper and 0.5% each of manganese and magnesium. The stronger tempers of this alloy are obtained by a solution heat treatment consisting of heating at 945° F for a short period depending on the thickness of the material and immediatley quenching in cold water. The material ages naturally at room temperature so that the full T4 temper is reached in about 4 days.

METHODS OF TESTING

Round End Tests.—Specimen 1 of each type was tested under static load as a column with round ends. The steel end fixtures bore against testing machine platens that were supported on knife-edge bearings as shown in Fig. 1. This loading condition corresponds to the attachment of a web member to a chord zero torsional resistance. The tubing specimens and the specimens of type H were centered on the platens so that the axis of the specimen was in the plane of the knife-edge bearings of the platens. Thus, the shafts of the mem-

bers were loaded axially although the end connection for specimen A1 (specimen 1, type A) was subjected to some bending. The single-angle specimens were placed so that the plane of shear between the angle and the gusset plate was in the plane of the knife-edges.

Deflections at the center were measured with a scale and taut wire. In some cases the deflections at points near the gussets were also measured.

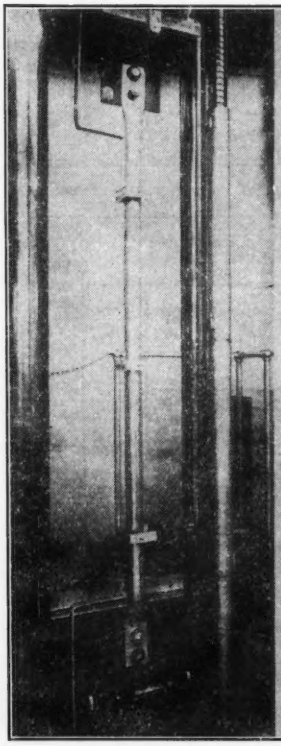


FIG. 1.—SPECIMEN A1; SETUP FOR TESTING A COLUMN WITH ROUND ENDS (KNIFE-EDGED BEARING FIXTURES)

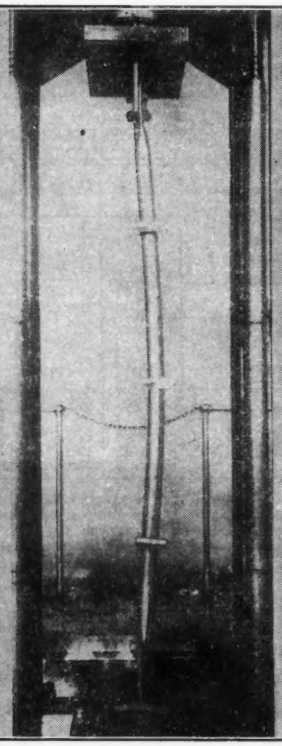


FIG. 2.—SPECIMEN A3 UNDER LOAD AS A COLUMN WITH FLAT ENDS

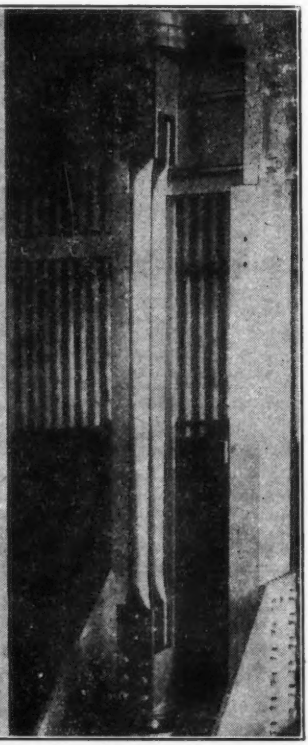


FIG. 3.—SPECIMEN A5; TWO-PART SPECIMEN AND ADAPTERS FOR TEST UNDER REPEATED LOAD

The static load tests were made in an Amsler testing machine of 300,000-lb maximum capacity and intermediate load ranges. The load range of 30,000 lb was used in these tests. The machine (shown in Figs. 1 and 2) is of the four-column type and, being relatively rigid, is well adapted for column testing. Two specimens of 52S–H32 round tubing (specimen A1 with round ends and specimen A3 with flat ends) are shown under load. The periodic calibrations indicate that the loads are correct within 0.5% for the load range used.

Flat End Tests.—Two specimens of each type but of different lengths were tested under static load as columns with flat ends. The steel end fixtures bore against testing machine platens that were fixed against tipping and rotation

(see Fig. 2). This condition corresponded to the attachment of a web member to a chord of great torsional stiffness. Deflections at the center were measured with a scale and taut wire, and strains were measured at a number of points by type A Huggenberger tensometers (mechanical multiplication of about 1,200). At points of rapidly changing stress, a gage length of 0.5 in. was used; at other places a gage length of 1 in. was used.

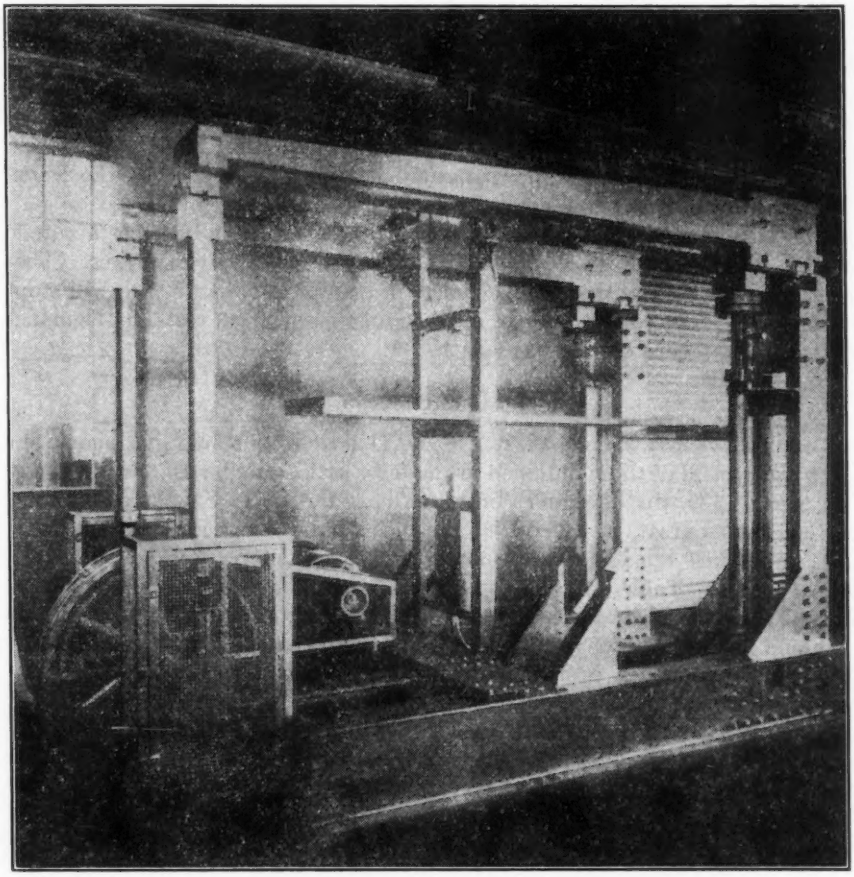


FIG. 4.—LABORATORY SETUP FOR TESTING COLUMN FATIGUE

Fatigue Tests.—Units consisting of a pair of tubes or angles 61 in. long were tested under repeated load ranging from zero to a maximum in compression. The two-part specimen was attached to the adapter of the testing machine as shown in Fig. 3. Using plate fulcra within the adapters the total load was divided equally between the two members. After the load range has been adjusted, the machine was operated by hand while strains were measured at the center and near one end, again using the type A Huggenberger tensometers.

The relative deflection of the two members was measured with calipers and a scale.

The fatigue tests were made in units 3 and 4 of the structural fatigue testing machines[10] of the Aluminum Research Laboratories shown in Fig. 4. Essentially each unit consists of two parallel beams, one the foundation beam and the other the loading beam. They are hinged together by plate fulcra at one end and by an adjustable-throw motor-driven crank at the other. The specimen is inserted between the two beams near the hinge end and undergoes a cyclic loading as the crank rotates. The loading beam is used as the load weighing device and the deflection of the beam is measured while the crank is turned slowly by hand. The load on the specimen is then obtained from the calibration curve previously determined. The inertia effects for the machine operating at normal speed (210 cycles per min) have been determined so that the necessary small correction (approximately 2%) can be applied to the loads obtained from the measured deflections of the loading beam.

The machine is equipped with a cycle counter and each unit has a cutoff switch. The machine is stopped automatically when a specimen fails. The arrangement is so sensitive that a change in load of less than 600 lb will stop the machine. The construction of the machines is such that tests can be made readily on specimens of only one length.

Discussion of Results

Column Strengths in Static Tests.—The load deflection relations shown in Fig. 5 are typical of those obtained in the tests with static loads. In the tests with round ends, the deflections of points near the ends of the gusset plates were almost as great, in general, as the deflection at the center, so that the tubes and angles remained almost straight and the failures were of a double-hinge type (see Table 4). The specimens appeared to be articulated with joints at the knife-edge bearings and at the gusset plate or flattening of the shaft. As should be expected for members with this characteristic, the ultimate loads were relatively low. A comparison of the test results with the Euler column curve is of interest, but it should be borne in mind that with such discontinuities along the length and eccentricities of loading these specimens do not fulfil the conditions of an Euler column. The comparison is shown in Fig. 6, where the data points are plotted at slenderness ratios corresponding to a length equal to the distance between knife-edges and the radius of gyration about the axis parallel to the gusset plate in the case of the symmetrical sections or the axis of least stiffness in the case of the single angle. From this comparison, it appears that the type A specimen was most nearly like an Euler column since it supported 76% of the computed load. The connection at the end fixture was probably much stiffer in this specimen than in any of the others since it was attached directly to the steel end fixtures. Specimen E1 which had the greatest eccentricity on the angle was supporting only 34% of the Euler load (gross area) when the test was stopped because of the large deflections. The test on specimen G1 was also stopped for this reason, but in this case the load was 51% of the Euler load. Although specimen H1 has a symmetrical section,

[10] "Fatigue Machines for Testing Structural Units," by R. L. Templin, *Proceedings*, A.S.T.M., Vol. 39, 1939, pp. 711–722.

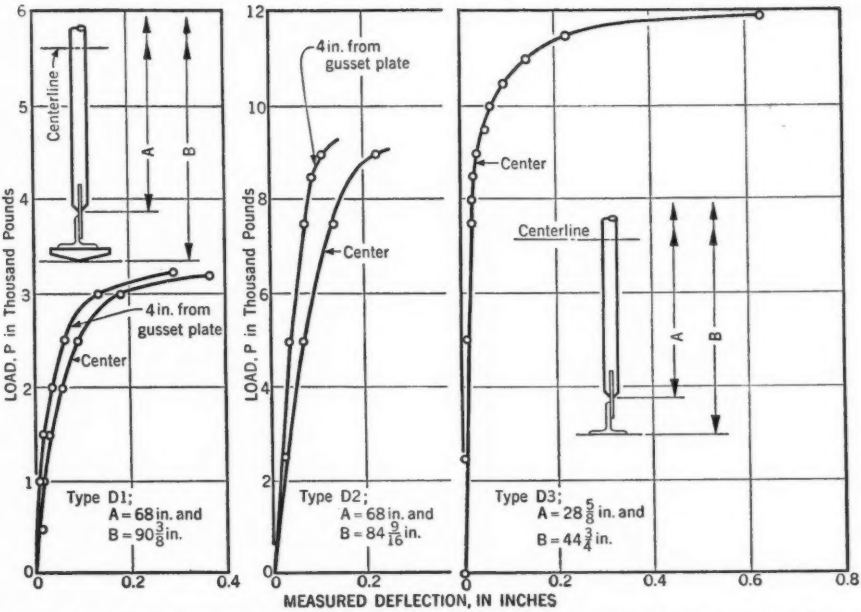

LOAD, P in Thousand Pounds
MEASURED DEFLECTION, IN INCHES
Centerline
A
B
4 in. from gusset plate
Center
Type D1; A = 68 in. and B = 90 3/8 in.
Type D2; A = 68 in. and B = 84 9/16 in.
Type D3; A = 28 5/8 in. and B = 44 3/4 in.

FIG. 5.—LOAD DEFLECTION CURVES OF STRUTS LOADED THROUGH GUSSET PLATES (TUBING DIMENSIONS: OUTSIDE DIAMETER, 2.5 IN.; WALL THICKNESS, 0.134 IN.; AND SECTION AREA, 0.904 SQ IN.)

TABLE 4.—RESULTS OF STATIC TESTS ON STRUTS LOADED THROUGH GUSSET PLATES

Specimen (1)	Description of specimen (2)	Over-all length (in.) (3)	Maximum load, P (lb) (4)	Type of failure and test condition (5)
A1	Flattened end tube	88 11/16 [a]	4,920	Double hinge at ends.[b,c]
A2		82 3/8	11,600	Lateral deflection.[b,d]
A3		43 9/16	19,390	Lateral deflection.[b,d]
B1	Tube flattened and split	89 1/4 [a]	1,540	Double hinge at ends.[c]
B2		84 9/16	9,550	Double hinge at ends.[d]
B3		43 1/16	10,930	Double hinge at ends.[d]
C1	Tube split but not flattened	90 3/8 [a]	2,800	Double hinge at ends.[c]
C2		84 9/16	12,250	Lateral deflection.[d]
C3		44 1/8	18,600	Lateral deflection.[d]
D1	Tube split, welded, but not flattened	90 3/8 [a]	3,190	Double hinge at ends.[c]
D2		84 9/16	9,400	Lateral deflection, hinged at gussets.[d]
D3		44 1/8	11,900	Lateral deflection, hinged at gussets.[d]
E1	Plain angle	90 3/8 [a]	3,500[e]	Double hinges at ends.[c]
E2		84 9/16	16,050	Lateral deflection and twisting.[d]
E3		45	16,400	Rivets sheared.[d]
E4		82 13/16	18,400	Lateral deflection and twisting.[b,d]
E5		43 9/16	29,425	Lateral deflection and twisting.[b,d]
F1	Flattened angle	90 11/16 [a]	1,500	Double hinge at ends.[c]
F2		84 1/2	20,150	Lateral deflection.[d]
F3		45 1/8	17,075	Lateral deflection.[d]
G1	Half-flattened angle	90 1/16 [a]	5,300[e]	Double hinge at ends.[c]
G2		84 1/4	18,570	Lateral deflection.[d]
G3		44 1/4	24,850	Lateral deflection.[d]
H1	Double angle	90 [a]	2,000	Double hinge.[c]
H2		84 1/2	20,800	Gusset plate buckled.[d]
H3		82 13/16	23,500	Lateral deflection.[b,d]

[a] Between knife-edges. [b] Attached directly (by turned studs) to steel end fixture. [c] Tested as a column with round ends, with knife-edge fixtures. [d] Tested as a column with flat ends. [e] Test stopped because of large deflections.

it supported only 15% of the Euler load, probably because the relatively thin gusset plate made a rather effective hinge. In specimen F1, the secondary hinge was at the ends of the flattened part of the angle where the material was only $\frac{1}{4}$ in. thick, as compared to the $\frac{1}{2}$-in.-thick gusset plate. A comparison of the strengths of specimens F1 and G1 indicates the undesirability of com-

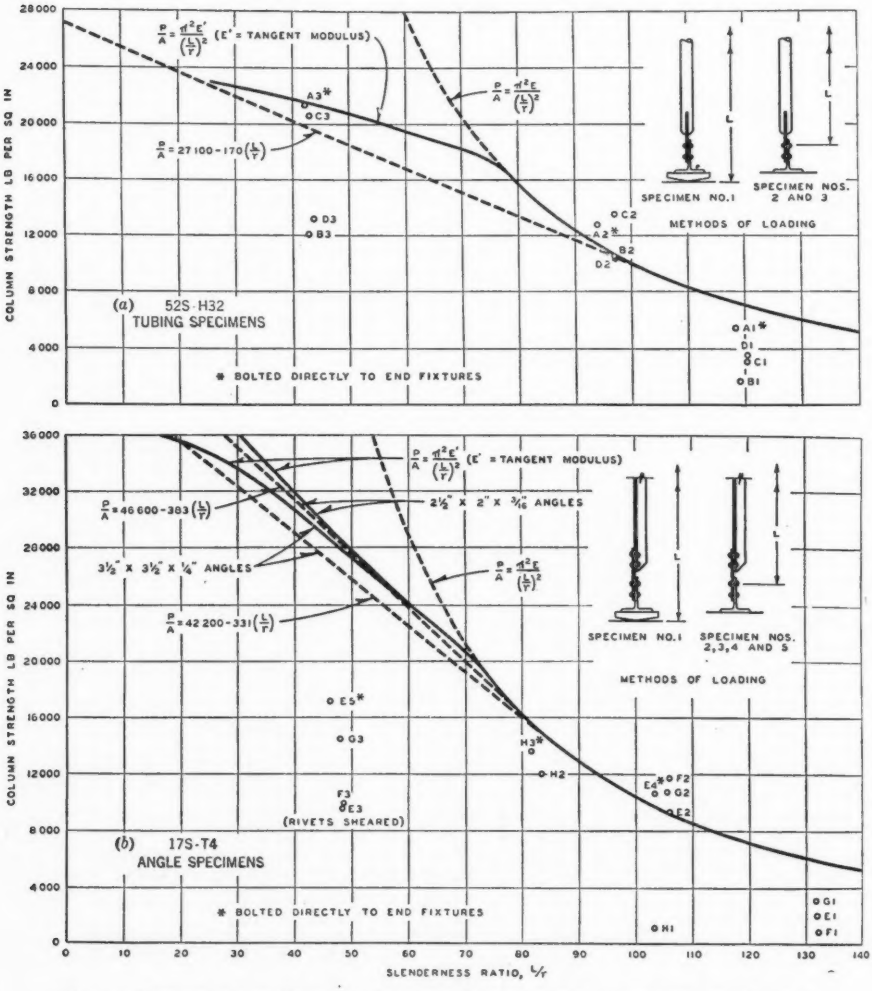


FIG. 6.—COMPARISON OF TEST RESULTS WITH COMPUTED CURVES OF COLUMN STRENGTH

pletely flattening the angle. In other words, it is desirable to maintain some of the stiffness of the member in the region of the connection.

Figs. 2 and 7 show typical failures of the longer specimens tested as columns with flat ends and Fig. 8 shows the failure of the short specimens of types C and E.

In Fig. 2 it will be noticed that the tube appears to be hinged near the end of the flattened part. The round end Euler column load for a tube with the length equal to the distance between the inside studs (70.5 in.) is 10,300 lb. The test result is 11,600 lb—showing some restraint, as should be expected.

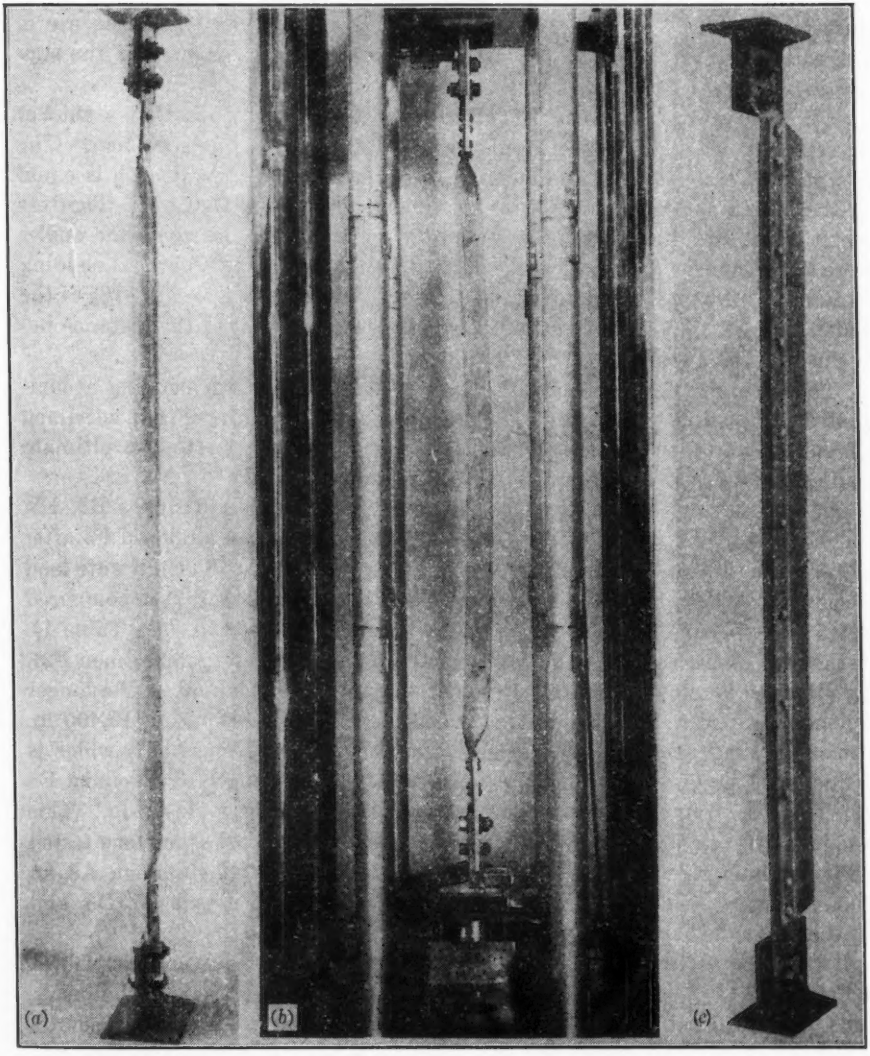


FIG. 7.—LONG SPECIMENS, AFTER FAILURE AS COLUMNS WITH FLAT ENDS: (*a*) SPECIMEN B2 (52S–H32 TUBING); (*b*) SPECIMEN F2 (17S–T4, EQUAL-LEG ANGLE); AND (*c*) SPECIMEN H2 (17S–T4, DOUBLE ANGLES)

The indicated fixity coefficient K is about 0.94. With less rigid end fixtures the degree of restraint, of course, would have been less.

Specimen B2 shown in Fig. 7(*a*) exhibits a double-hinge type of failure similar to that exhibited by the specimens tested with the knife-edge fixtures.

Of course, in this case both hinges developed within the specimen, one at the end of the flattening and one in the gusset plate. Specimen F2 (see Fig. 7(*b*)) with a thicker gusset plate ($\frac{1}{2}$ in. as compared to $\frac{1}{4}$ in.) did not develop a double-hinge effect although the same two points of weakness were present. Fig. 7(*b*) shows this specimen under a load near the maximum. The bending of the gusset plates is apparent. It appears from this comparison that the use of rather thick gusset plates is advisable with members of this type, if the supporting structure can furnish some restraint.

Fig. 7(*c*) shows specimen H2 after testing. It is obvious that a thicker gusset plate would have increased the ultimate load of this specimen. The computed Euler load for this member, assuming the effective length is equal to the length between inside studs of the end fixtures, is 25,900 lb. The test result is 80% of this value. The failure of specimen H3, in which the angles were attached directly to the steel end fixtures (Table 2), was by lateral bending parallel to the plane of the steel gusset plates. The ultimate load is 87% of the Euler load for a specimen having an effective length equal to the distance between the inside studs.

The plain-angle specimen (E2, Table 2) failed by lateral bending accompanied by twisting. Specimen E4 (which was about 9 in. longer than specimen E2 and was attached directly to the steel end fixture) supported an ultimate load approximately 15% greater than that of specimen E2.

Of the shorter specimens tested with flat ends all but specimens B3, E3, and F3 failed by lateral deflection. The shapes of specimens B3 and F3 after failure gave the appearance of the double hinge at one end. The ultimate load of specimen B3 (which had a tube length of $28\frac{5}{8}$ in.) was 10,930 lb as compared to 9,550 lb for specimen B2 which had a tube length of 68 in. (see Table 1). In the case of the flattened angles (specimen F3 as compared to specimen F2), the ultimate load of the shorter specimen was only 85% of that of the longer specimen. Specimen E3 failed by shearing the rivets at a load of 16,400 lb. The single shear stress on the 53S–T4 rivets was 26,800 lb per sq in. which is in line with the expected value for such rivets. The strength of specimen E5 which had the same over-all length but no gusset plate was 29,425 lb. This specimen after failure is shown in Fig. 8(*b*). The other short specimens tested with flat ends failed by lateral deflection. In the case of specimen A3 an apparent hinge developed in the flattened part, and in specimens C3, D3, and G3 the hinge developed in the gusset plates.

The column strengths developed in the tests with flat ends are also shown in Fig. 6. The effective length was taken as the distance between the inside studs of the end fixtures. In addition to the Euler column curve, Fig. 6 shows curves of column strength as computed by the reduced-modulus and straight-line formulas for the tubing and angles, respectively. The reduced-modulus formula is the same as the Euler formula, except that in this comparison the tangent modulus is used as the reduced modulus for values of average stress above the proportional limit. The straight-line formula for the 52S–H32 tubing, based on the yield strength given in Table 3(*b*) and determined by a

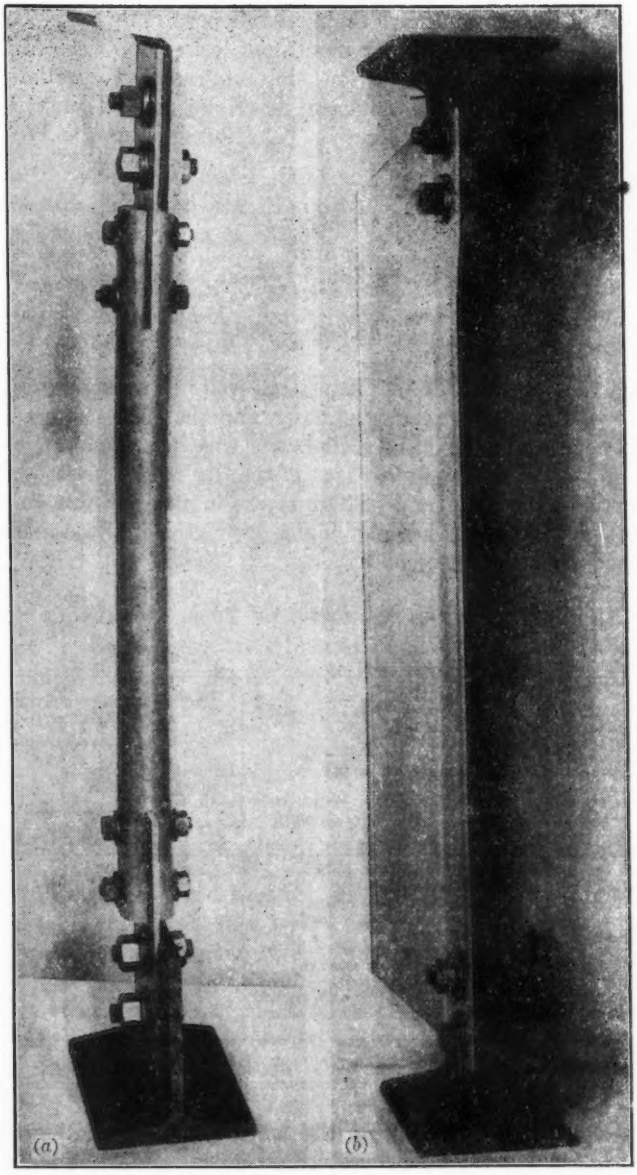


FIG. 8.—SHORT SPECIMENS, AFTER FAILURE AS COLUMNS WITH FLAT ENDS: (*a*) SPECIMEN C3 (52S–H32 ROUND TUBING) AND (*b*) SPECIMEN E5 (17S–T4, EQUAL-LEG ANGLE)

procedure developed in 1938, is as follows:[11]

$$\frac{P}{A} = 27{,}100 - 170\,\frac{K\,L}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

[11] "Column Strength of Various Aluminum Alloys," by R. L. Templin, R. G. Sturm, E. C. Hartmann, and Marshall Holt, *Technical Paper No. 1*, Aluminum Research Laboratories, New Kensington, Pa., 1938.

in which P/A is the column strength, in pounds per square inch; L is the length, in inches (between knife-edges for round end tests or between inside studs of end fixtures for flat end tests); r is the minimum radius of gyration of the cross section, in inches; and K is the coefficient describing the end conditions, taken as unity for the aforementioned lengths. The formula for the 17S–T4 alloy,[11] $3\frac{1}{2}$-in. by $3\frac{1}{2}$-in. by $\frac{1}{4}$-in. angles, is

$$\frac{P}{A} = 42{,}200 - 331\frac{KL}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and the formula for the $2\frac{1}{2}$-in. by 2-in. by $\frac{3}{16}$-in. angles[11] is

$$\frac{P}{A} = 46{,}600 - 383\frac{KL}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The fact that most of the test values fall well below the curves reflects the uncertainty of the effective lengths chosen and the effects of eccentricities in loading. The effect of the initial eccentricity was magnified in the cases of the double-hinge type of failure by the lack of rigidity in the gusset plates and the flattened ends of the members. The increase in strength due to rigid gusset plates is proved by the consistently higher strengths of those members which were fastened directly to the end fixtures.

TABLE 5.—RELATIVE STRENGTHS OF TUBE AND ANGLE STRUTS UNDER STATIC LOADING

Type	Form	Description of end	Thickness of gusset plates (in.)	Relative Average Stress at Failure (Gross Area)[a]		
				Round Ends	Flat Ends	
				L = 90 in.	L = 90 in.	L = 31 in.
A	Tube	Flattened, not slotted, bolted	None	100	100	100
B	Tube	Flattened, slotted, riveted	$\frac{1}{4}$	31	82	57
C	Tube	Not flattened, slotted, bolted	$\frac{1}{4}$	57	106	96
D	Tube	Not flattened, slotted, welded	$\frac{1}{4}$	65	81	62
E	Single angle	Riveted by one leg	$\frac{1}{2}$	38	73	45[b]
	Single angle	Bolted by one leg	None	....	84	80
F	Single angle	Completely flattened, riveted	$\frac{1}{2}$	16	92	47
G	Single	Partly flattened, riveted	$\frac{1}{2}$	57	84	68
H	Double angle	Symmetrical about one axis, riveted	$\frac{1}{4}$	21	94	....
	Double angle	Symmetrical about one axis, bolted	None	....	106	....

[a] Values of L are approximate. [b] Rivets sheared.

Fig. 6 also shows that, except for the tubular types A and C, the strengths of the short flat-end struts were considerably less in proportion to the strengths predicted by the formulas than were the strengths of the longer specimens. This fact can also be explained partly by the effect of eccentric loading. The initial eccentricity of a given type of connection is a constant, of course, regardless of the length of the member. If the member then fails by the formation of the double hinges at the ends with the central part of the member remaining comparatively straight (as in the case of specimens B3 and F3), the strength is essentially independent of the length.

A further comparison of the test results can be found in Table 5, in which the relative static column strengths are listed. The values are based on average stress at failure. The strengths of the two, type E, plain angle, specimens which were attached directly to the end fixtures, are seen to be 84% and 80% of the strengths of the corresponding type A tubes flattened over filler tubes and attached directly to the end fixtures. Of the specimens attached to the end fixtures through gusset plates, the type C tubes with segmental filler plugs show the most consistently high strength.

TABLE 6.—SUMMARY OF STRESSES MEASURED IN FLAT END TESTS

(Stresses Are Compressive Unless Marked +; All Stresses Are in Pounds per Square Inch)

Specimen	Over-all length (in.)	Gross area (sq in.)	Load[a] (lb)	Average Stress (Lb per Sq In.)		Range of Stress at:				Lateral deflection at center line (in.)	Ratios	
						Center Line		End				
				Computed P/A	Measured at center line	Minimum	Maximum f_m	Minimum	Maximum f_o		$\frac{f_o}{f_{avg}}$	$\frac{f_o}{f_m}$
A3....	43 $\frac{9}{16}$	0.904	9,500	10,510	10,520	9,380	11,640	+ 2,850	18,460	0.034	1.75	1.58
B2....	84 $\frac{9}{16}$	0.904	9,500	10,510	10,740	7,600	14,160	+ 1,440	20,690	0.294	1.92	1.46
B3....	43 $\frac{13}{16}$	0.904	9,500	10,510	10,320	9,380	11,750	+ 2,030	28,380	....	2.74	2.41
C2....	84 $\frac{9}{16}$	0.904	7,500	8,300	7,640	250	15,200	5,540	10,560	....	1.38	0.70
C3....	44 $\frac{1}{2}$	0.904	7,500	8,300	8,670	8,410	8,820	6,210	10,300	....	1.19	1.17
D2....	84 $\frac{9}{16}$	0.904	7,500	8,300	8,245	6,290	9,980	4,730	9,970	0.130	1.21	1.00
D3....	44 $\frac{3}{4}$	0.904	7,500	8,300	8,075	7,130	9,230	4,560	9,970	0.024	1.24	1.08
E2....	84 $\frac{9}{16}$	1.714	9,000	5,250	5,275	+9,800	11,490	+ 5,240	12,330	0.40	2.34	1.07
E3....	45	1.714	9,000	5,250	4,985	+5,860	12,740	+ 6,230	15,590	0.096	3.12	1.22
E4....	82 $\frac{13}{16}$	1.714	9,000	5,250	5,440	+3,120	9,780	+ 1,990	9,870	0.260	1.81	1.01
E5....	43 $\frac{9}{16}$	1.714	9,000	5,250	5,415	+4,350	10,470	+ 4,170	11,860	0.090	2.19	1.13
F2....	84 $\frac{1}{2}$	1.714	9,000	5,250	5,340	2,520	8,300	+13,950	17,220	0.032	3.22	2.35
F3....	45 $\frac{1}{2}$	1.714	9,000	5,250	4,990	1,530	8,190	+ 9,390	12,950	0.021	2.60	1.58
G2....	84 $\frac{1}{2}$	1.714	9,000	5,250	5,490	2,470	8,360	+10,360	16,740	0.134	3.05	2.00
G3....	44 $\frac{1}{2}$	1.714	9,000	5,250	5,335	2,100	7,650	+11,060	18,040	0.042	4.16	2.36
H3....	82 $\frac{13}{16}$	1.726	12,000	6,950	6,920	6,080	7,760	4,260	9,540	0.030	1.38	1.23

[a] Load at which strain measurements were made.

It should be remembered that the gusset plates for the single-angle specimens were $\frac{1}{2}$ in. thick, whereas $\frac{1}{4}$-in. gussets were used for the tubular and double-angle struts. Obviously the thinner gusset plate should be the more effective hinge. Weakness in the gusset plate probably accounts for the low comparative strength of the type H double-angle specimen in the round end test. On the other hand, in the flat end tests double angles proved to be superior to single angles, despite the failure of the gusset plate of specimen H2.

The data in Table 5 indicate a generally greater structural efficiency for the tubular and double-angle types of struts with relatively rigid end connections when compared with the single-angle struts.

Measured Stresses, Static Tests.—The measured strains were interpreted into stresses by simply multiplying by the modulus of elasticity;[12] thus, for 52S, E = 10,200,000 lb per sq in.; and, for 17S, E = 10,600,000 lb per sq in. The measured stresses thus obtained are summarized in Table 6 and specific examples are shown in Fig. 9. The dotted lines indicate the value of the

[12] "The Elastic Constants for Wrought Aluminum Alloys," by R. L. Templin and E. C. Hartmann, *Technical Report No. 966*, National Advisory Committee for Aeronautics, Washington, D. C., January, 1945.

average stress on the gross section of a strut. Compressive stress is plotted to the right and tensile stress to the left. The loads at which the measurements were made corresponded to average stresses ranging from 5,250 lb per sq in. for the single angles to 10, 510 lb per sq in. for the tubes of types A and B. Although there is considerable variation in measured stresses, the averages of the stresses measured at the center agree very well with the simple (P/A)-stresses. In thirteen of the seventeen specimens the average measured stress is within 5% of the computed average stress and only in the case of specimen

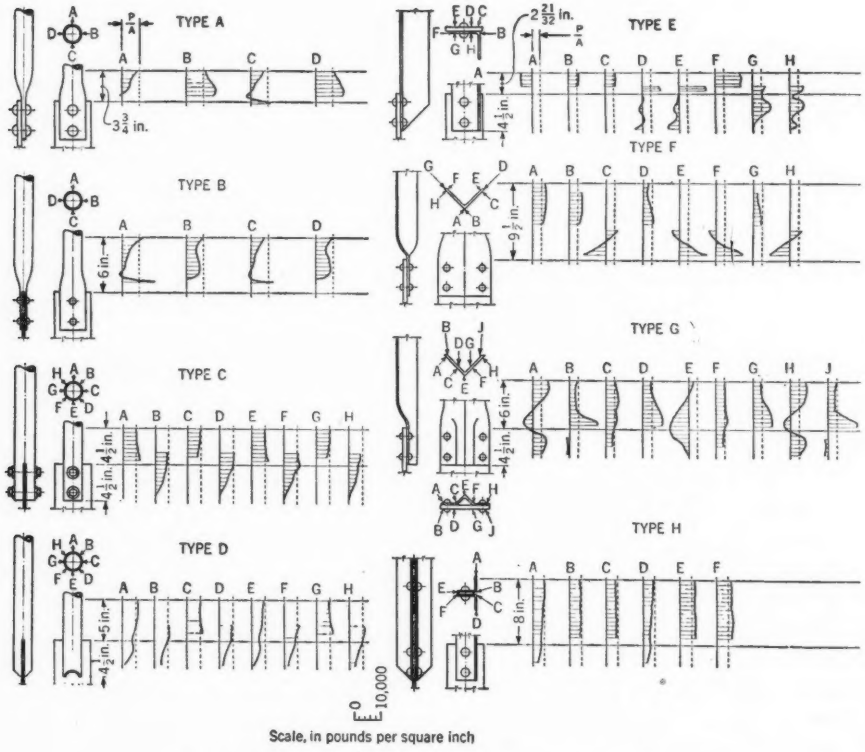


FIG. 9.—TYPICAL MEASURED STRESS DISTRIBUTIONS IN STRUTS

G3 is the difference greater than 8%. In general the range of stress at the end is greater than that at the center, and for the specimens with flattened ends it is much greater. The maximum compressive stresses are also generally greater at the end than at the center. For example, in the case of specimen A3 under a load of 9,500 lb, the stresses at the center range from 9,380 lb per sq in. to 11,640 lb per sq in. in compression whereas at the end the stresses range from 2,850 lb per sq in. in tension to 18,460 lb per sq in. in compression. The maximum compressive stress at the end is 1.6 times the maximum compressive stress at the center. Because of large lateral deflections, the range

of measured stresses at the center was greater than at the end for specimens C2, E2, and E4.

A study of Fig. 9 indicates that, in general, the stress distribution in the flattened ends of the tubes is alike, despite the difference in the attachment to the gusset plate. It can be noted that a tensile stress was developed in the tapered part of the tube near the flattening. The stress gradients in this region are rather high. For specimens of types C and D the stress gradients are relatively flat.

In the plain-angle specimens (type E) the load is eccentrically applied, but in addition to the eccentricity there are certain restraining moments that decrease the effects of the eccentricity. The restraining moment in one direction rises by virtue of the bending stiffness of the gusset plate and in the other direction by the use of more than one rivet in the connection. By working backward from the measured stresses, the actual eccentricities in the flat end columns were found to be, approximately: e_1 normal to the gusset plate = 0.75 in.; and e_2 parallel to the gusset plate = 0.31 in. to 0.60 in. The nominal eccentricity taken equal to the distance from the back of the angle to the centroid of the section is 0.94 in. The stiffness of the gusset plate accounts for the difference.

The eccentricities of single-angle connections were greatly reduced by the flattened ends of types F and G. The measured stresses indicated an effective eccentricity normal to the gusset plate of approximately 0.09 in. toward the heel of the angle in type F, and of 0.16 toward the toes of the angle in type G with the ends only partly flattened. Thus, stresses measured at the center of the specimens of types F and G were all compressive stresses, whereas at the center of specimen E2 the stresses varied from 11,490 lb per sq in. in compression to 9,800 lb per sq in. in tension. It appears that a greater amount of flattening of the G-type specimens would have reduced the eccentricity.

As in the case of the flattened end tubes, deforming the ends of the angles produced high stress gradients near the connections. Both maximum compressive and maximum tensile stresses measured at the ends of the specimens of types F and G are higher than those found in the specimens of type E.

Placing two angles back to back to make the strut symmetrical about one axis greatly reduced the stress variation. The range of stress at the center of specimen H3 was only from 6,080 lb per sq in. to 7,760 lb per sq in. in compression.

Results of Fatigue Tests.—Stress distributions in the fatigue specimens were found to be quite similar to the distributions found in the static tests of flat end specimens shown in Fig. 9.

In the case of the angles, two specimens of each type were tested in fatigue, one with the same average stress as that imposed on the tubing (approximately 7,500 lb per sq in.) and the other with the same total load. In the second specimen of each type the stress cycle was from zero to approximately 4,000 lb per sq in.

Table 7 summarizes the results of the fatigue tests giving ratios of measured stresses and the number of cycles of compressive loading imposed on

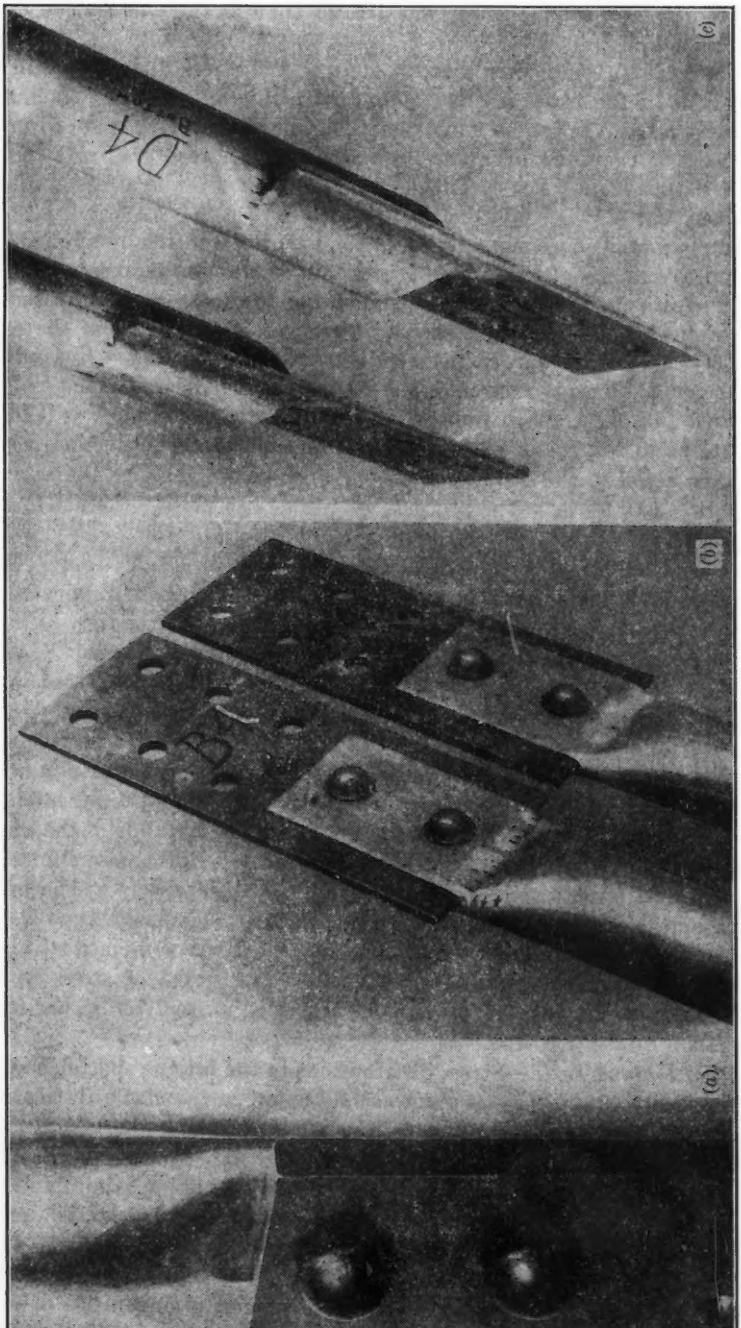


FIG. 10.—ENDS OF TWO-PART SPECIMENS, 52S-H32 ROUND TUBES, AFTER FAILURE UNDER REPEATED COMPRESSIVE LOAD: (*a*) BOTTOM OF SPECIMEN A5; (*b*) TOP OF SPECIMEN B4; AND (*c*) BOTTOM OF SPECIMEN D4

each specimen. Three of the specimens—C4, F5, and H5—withstood approximately 25,000,000 cycles of stress without visible signs of failure. For the other specimens, the number of cycles to produce visible cracks ranged as low as 71,900. In this case (specimen E6), failure occurred by shearing the rivets on which the average shear stress was about 20,400 lb per sq in. The failure of specimen F4 also involved the failure of a rivet. The average shear stress on the rivet was about 15,300 lb per sq in., and the number of cycles to failure

TABLE 7.—RESULTS OF TESTS ON COLUMN FATIGUE SPECIMENS; STUDY OF END CONNECTIONS FOR STRUTS

(Stresses Are Compressive Unless Marked +)

Specimen	Gross area (sq in.)	Load[a] P (lb)	Stress (lb per sq in.)				No. of cycles	Remarks
			Average	Maximum				
			P/A	Measured at Center Line		Measured at end		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
A4....	0.904[b]	13,650	7,550	8,000	12,700	−16,080 + 3,610	342,200	....
B4....	0.904[b]	13,520	7,480	7,710	9,070	10,700	172,000	Removed at 549,800.
C4....	0.904[b]	13,700	7,580	8,110	9,180	9,180	25,004,500	No failure.
D4....	0.904[b]	13,600	7,530	7,610	8,650	8,690	22,891,100	Small cracks at ends of gusset plates; at 25,208,800 cycles crack extended over three fourths of the circumference.
D5....	0.904[b]	13,760	7,620	7,690	8,390	9,130[c]	20,785,400	Small cracks at ends of gusset plates.
E6....	1.714[b]	26,370	7,690	7,550	−17,570 +14,510	−13,930	71,900	Rivets sheared.
E7....	1.714[b]	13,680	3,990	4,060	− 8,140 + 5,110	− 2,300 + 6,130	4,276,700	....
F4....	1.714[b]	25,630	7,480	7,740	14,530	.... +13,700	1,118,600	One rivet sheared, crack in angle.
F5....	1.714[b]	13,530	3,950	4,120	7,450	.... + 6,500	25,489,300	No failure.
G4....	1.714[b]	25,610	7,480	7,000	13,340	−12,000 +12,170	443,800	..
G5....	1.714[b]	13,640	3,980	3,860	5,970	− 6,230 + 7,100	5,508,200	....
H4....	1.726[b]	26,600	7,150	6,910	11,300	8,520	4,216,800	....
H5....	1.726[b]	13,580	3,940	3,820	6,080	4,950	24,899,100	No failure.

[a] Maximum load of stress cycle and load at which strain measurements were made. [b] Two each. [c] Stresses at ends of diameter parallel to the gusset plate average 9,040 lb per sq in. as compared to an average of 7,888 lb per sq in. for the two diameters at an angle of 45° to the gusset plate. In specimen D4, the averages for all the diameters were practically the same.

was 1,118,600. These two test results lie well above the (S-N)-curve indicated by the data for joints with cold driven 53S–T4 rivets reported by E. C. Hartmann, M. ASCE, J. O. Lyst, and H. J. Andrews, Jun. ASCE, in 1944.[13] This difference is to be expected since the riveted joints failed by tension in the plates whereas in these specimens the failures were in the rivets.

The test results in Table 7 further indicate the superiority of symmetrical connections over unsymmetrical connections and show that flattening the ends

[13] "Fatigue Tests of Riveted Joints, Progress Report of Tests of 17S–T and 53S–T Joints," by E. C. Hartmann, J. O. Lyst, and H. J. Andrews, *Advance Restricted Report No. 4115*, National Advisory Committee for Aeronautics, Washington, D. C., September, 1944.

of tubular members decreases their resistance to repeated loads. Photographs of some of the specimens are shown in Figs. 10 and 11. Because of the local nature of the fatigue failures, it is to be expected that the strengths and lives of the specimens were little affected by the length of the specimen.

General Remarks.—The results of the tests described in this paper corroborate the general conclusion of J. A. Van den Broek,[14] M. ASCE, based on compressive tests of single angles with different types of connections—namely, that the compressive strength of angles is affected largely by the eccentricity

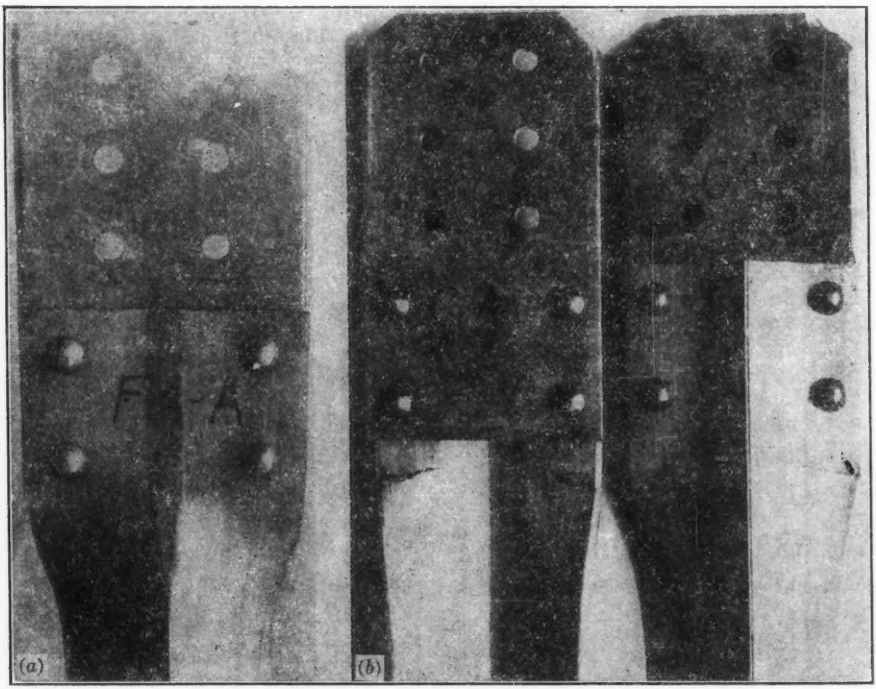


FIG. 11.—TOP ENDS OF TWO-PART SPECIMENS 17S-T4, EQUAL-LEG ANGLE, AFTER FAILURE UNDER REPEATED COMPRESSIVE LOAD: (*a*) PART A, OF SPECIMEN F4 AND (*b*) SPECIMEN G4

of the applied loads. Although Mr. Van den Broek found that angles with completely flattened ends showed greater eccentricity and less column strength than angles connected by only one leg, the opposite effect was found in the tests described in this paper. Since the eccentricity of the connections of a flattened end angle depends on the manner in which the angle is flattened and the face to which the supporting fixtures are attached, this comparison cannot be further analyzed because the details of the end connections are not discussed in Mr. Van den Broek's paper.

[14] "Effect of Connections and Rivet Holes on Ductility and Strength of Steel Angles," by J. A. Van den Broek, *Civil Engineering*, February, 1940, p. 83.

Various types of welded connections for tubular struts were described by A. Quisling and C. M. Cosman,[6] in 1943. With reference to flattening the ends of tubes they conclude that tubes with flattened ends

> "* * * suffer from the weakness inherent in drastic changes of the shapes of the members, always objectionable. The majority of them are also less rigid laterally, lacking the stabilizing effect of the tubular shape at the joints, where it is most needed."

These statements seem to be borne out by the results of these tests at the Aluminum Research Laboratories. However, the comparatively high static strengths of the type A flattened end tubes (Table 1) bolted directly to the end fixtures indicate that the loss in ridigity due to flattening the ends may be somewhat lessened by first inserting a filler tube to reduce the degree of flattening and by using very rigid gusset plates.

Concluding Remarks

Although the tests made for evaluating the various types of end connections were made on aluminum alloy members, it is believed that the general conclusions drawn are applicable to members of other materials.

1. Round tubes are superior in static column strength to single angles, provided that the end connections are sufficiently rigid. Of the types of specimens that were fastened to the end fixtures through gusset plates, the type C unflattened tubes, slotted and bolted through segmental filler plugs, showed the most consistently high strength (see Table 5). Based on the average stresses at failure for the types A and E specimens which were attached by turned studs directly to the end fixtures, the static strength of round tubes of type A, with the ends flattened over filler tubes, was about 20% greater than the strength of single angles of the same length, fastened through one leg.

2. Double-angle struts, symmetrical about one axis, are superior in static column strength to single angles, provided that the end connections are sufficiently rigid (see Table 6). The static strengths of the various long single-angle struts tested as columns with flat ends were from about 76% to 96% as great as the average strength of the long double-angle strut tested as a column with flat ends.

3. The stiffness of the gusset plates and the rigidity of the connections are important factors contributing to the static strength of struts.

4. The static strengths of the round tubes tested as columns with round ends were from 24% to 76% of the values computed by the Euler column formula taking the effective length as the length between knife-edges of the supporting fixtures. The corresponding range for the angles was from 15% to 51%.

5. The strengths of long struts of the types tested, loaded as columns with flat ends, ranged from 20% below to 27% above the values computed by the Euler formula using a fixity coefficient equal to unity and an effective length equal to the length between inside studs of the end fixtures.

6. The strengths of the short tubular struts tested as columns with flat ends ranged from 61% to 108% of the values computed by the straight-line column formula taking the effective length equal to the distance between inside studs of the end fixtures and a fixity coefficient equal to unity. The corresponding range for the single-angle struts was from 37% to 55%.

7. Of the types of specimens tested, double angles and round tubes bolted or welded to gusset plates placed in slots in the unflattened ends showed the most favorable distribution of stress near the connections. The ratios of maximum measured stress to average stress for the various types of connections were as follows:

Description of end connection	Ratio: $\frac{\text{Maximum stress}}{\text{Average stress}}$
Slotted tube, unflattened, or double angles	1.38
Flattened tube, not slotted, filler tubes in ends	1.75
Flattened tube, slotted	2.74
Single angle, fastened through one leg	3.12
Completely flattened angle	3.22
Partly flattened angle	4.16

The stresses were measured at loads ranging from 7,500 lb to 12,300 lb depending upon the type of specimen (see Table 6).

8. Of the types of tubular specimens tested, round tubes with the ends unflattened and bolted to gusset plates placed in slots in the ends showed the greatest resistance to repeated loads producing a stress cycle from zero to about 7,500 lb per sq in. in compression. Similar specimens welded rather than bolted to the gusset plates showed slightly less fatigue strength, whereas flattened-end tubes showed the lowest resistance to repeated stress.

The fatigue lives of the various tubing specimens tested were as follows (cycle of average stress from zero to about 7,500 lb per sq in. in compression):

Description of end connections	Number of cycles of stress
Slotted ends, bolts through segmental filler plugs	25,004,500 No failure
Slotted ends, welded to gusset plates	22,891,100 20,785,400
Flattened ends, not slotted, unsymmetrical connection	342,200
Flattened ends, slotted, symmetrical connection	172,000

9. Angles with the ends flattened in order to load the member close to the neutral axis showed greater fatigue strength than angles connected by one leg, but less fatigue strength than double angles symmetrical about one axis. The fatigue lives (number of cycles) of the various angle specimens tested, with

average compressive stresses from zero to about 4,000 lb per sq in. and 7,500 lb per sq in., respectively, were:

Description of end connections	0 to 4,000 lb per sq in.	0 to 7,500 lb per sq in.
Double angle, symmetrical connection	24,899,100 No failure	4,216,800
Completely flattened angle	25,489,300 No failure	1,118,600
Partly flattened angle	5,508,200	443,800
Single angle fastened through one leg	4,276,700	71,900 Rivet failure

10. A desirable combination of static strength and resistance to repeated loads is furnished by round tubes, unflattened, with gusset plates placed in slots in the ends, and by double angles, symmetrical about one axis.

11. Flattened end tubes are roughly comparable in static strength and resistance to fatigue to single angles fastened through one leg.

Acknowledgments

The work described in this paper was done at the Aluminum Research Laboratories under the direction of R. L. Templin, M. ASCE, assistant director of research and chief engineer of tests and of Mr. Hartmann, chief, Engineering Design Division. The paper was sponsored by the Committee on Design in Lightweight Structural Alloys of the ASCE Structural Division.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

PROPERTY SURVEYS MUST FIT THEIR TITLES

Discussion

BY WILLIAM C. WATTLES

WILLIAM C. WATTLES.[12]—Readers and discussers of this paper should realize that any rational enlargement of the subject matter, beyond the basic principles and generalities given, would produce a volume far beyond any reasonable scope. An example of this scope is afforded by the valuable work on this subject, presented by the late Ray H. Skelton, M. ASCE, in 1930.[13]

Each survey of land boundaries establishing the location of record title and relating the same to physical conditions presents a particular problem, different in most of its aspects from any other, except as to basic principles. These differences are responsible for many of the variances in legal decrees given in superficially similar cases.

Mr. Borg's claim that certain statements in the paper may be misleading, quoting legal citations to support his discussion, is not proved. The paper, taken as a whole, indicates the uncertainties present in any survey, and cautions against the flat acceptance of any principle by itself, but requires its analytical association with others. Item 8 ("Any other matters connected with the property in question") in the list defining (determining) boundaries, includes examination of any citations covering matters similar to the problem at hand.

The question of reliability or superiority of physical monuments over the description calls of record is a matter of investigation and analysis. If the surveyor is making a location from record, it is a general rule to give secondary value to physical monuments of whatever kind, as against the calls and non-physical legal monuments recited in the description. If the description calls for physical monuments, or if those existing on the ground can be sufficiently identified in position with the record, then the monuments will have superiority. A subsequent court decision, adverse to this rule, should be construed as a modification because of other factors in the case, and not as a failure of the rule.

Mr. Borg is correct in stating that the intent of the parties will finally

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NOTE.—This paper by William C. Wattles was published in February, 1948, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1949, by Robert F. Borg, and S. A. Bauer; June, 1949, by Rupert F. Carroll; and September, 1949, by Henry D. Southerland, Jr.

[12] Glendale, Calif.

[13] "The Legal Elements of Boundaries and Adjacent Properties," by Ray H. Skelton, Bobbs-Merrill Co., Indianapolis, Ind., 1930.

govern, but it is the court, not the surveyor, who may assume an intent beyond that specifically expressed in the record.

Mr. Bauer's discussion is an excellent and complete summary of the necesdary coordination between survey and title. Also, he points out the methods of attaining such coordination through education of the surveyor and layman to recognize facts as opposed to assumptions.

Mr. Carroll is to be complimented on his discussion of survey difficulties in Texas; it is a valuable addition to survey information. His statement regarding a knowledge of the practice of previous surveyors is very pertinent and applies elsewhere as well as in Texas. Often such knowledge marks the difference between correct and incorrect location.

It is impossible to answer fully Mr. Southerland's implied questions within the scope of this discussion, for he approaches the realm of original error in record titles and the corrective procedure thereof; this is not within the authority of the surveyor nor the province of the title location survey.

It is perfectly true that many descriptions are not "locatable" from their own statements or recitals and that such descriptions, if in a conveyance of any kind, cannot pass a perfect title; therefore, any survey under such conditions cannot fit title, but must be considered in the light of a fact-finding procedure for purposes of adjustment by legal methods.

It is recognized that surveys are made for many purposes other than property boundary relocation. In such cases the matter of relative values and accuracies may enter. A property survey in accord with the title must be as particular and exact as such title can be determined; any departure, such as assumption of possession boundaries and agreed lines not confirmed by record, converts the title location survey to another form.

The purpose of the paper was to present a series of principles of property boundary location founded on the record title as the best and most consistent base of operation. These principles must be associated or modified as need be, in any particular description or case. In the event of an inability to establish a correct technical position, because of an error or an insufficiency in the record, the problem resolves into a pure location for purposes of adjustment by legal methods.

Matters of cost, relative values of land and survey expense, mechanical technique, allowable errors, and the like, have no part in the analysis of property survey conformable with title.

The writer wishes to thank the discussers for their timely, pertinent, and friendly consideration of the paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

AIR-INLET VALVES FOR STEEL PIPE LINES

Discussion

BY W. E. EVANS

W. E. EVANS.[4]—The size of air-inlet valve required to maintain a certain internal pressure in a pipe line during transient flow conditions can be determined easily by the method proposed in this paper. Undoubtedly, there are many conditions that may exist in a pipe line to which these data would apply, and it is one of these special cases that will be described herein.

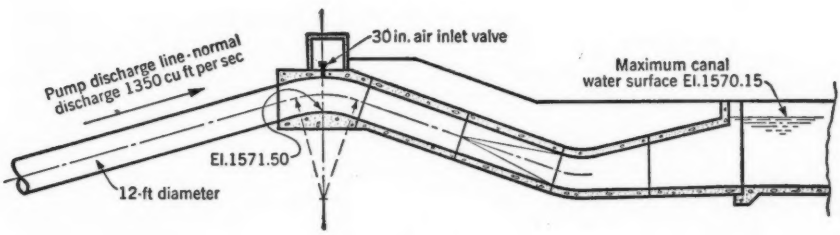
30 in. air inlet valve
Pump discharge line-normal discharge 1350 cu ft per sec
Maximum canal water surface El.1570.15
El.1571.50
12-ft diameter

FIG. 5

At the discharge outlet of the pumping plant pipe lines at Grand Coulee Dam in Washington there is a siphon as shown in Fig. 5. The problem of breaking the siphon action for return flow through the discharge line following power failure at the pumps arose. This problem was not associated with the collapse of the pipe line but rather with the loss of water from the canal by siphon action maintaining return flow through the pump.

The siphon head available was limited and depended upon the canal water surface elevation, relative to the siphon invert elevation. This relationship determined the internal pressure that had to be obtained and maintained in the discharge line to break the siphon action.

In addition to this internal pressure, the rate at which air must be supplied had to be determined. This aspect of the problem was handled by recourse to

NOTE.—This paper by John Parmakian was published in June, 1949, *Proceedings*.

[4] Engr., Branch of Design and Constr., Bureau of Reclamation, Denver, Colo.

the pump characteristics of the model pumps for Grand Coulee.[5] From these characteristics the steady-state reverse discharge at normal head was obtained as approximately 0.5 times normal pump discharge. Transient values of reverse discharge were higher than this steady-state value but were considered as too short-lived to be a basis for selecting the size of air valve needed.

Then, knowing the internal pressure necessary to prevent siphon action and expressing the steady-state reverse discharge in terms of velocity, the curves shown in Fig. 3 were used to obtain the size of the air value required. It was concluded that an air valve 30 in. in diameter, or its equivalent, was needed. This air valve would supply air at a rate equal to the steady-state reverse discharge through the pumps with the internal pressure in the pipe line at the valve adequate to break the siphon action.

[5] "Centrifugal-Pump Performance as Affected by Design Features," by R. T. Knapp, *Transactions*, A.S.M.E., Vol. 63, 1941, p. 251.

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DISCUSSIONS

SOME EFFECTS OF ANAEROBIC DIGESTION ON SEWAGE SLUDGE

Discussion

BY A M RAWN AND E. J. CANDELL

A M RAWN,[15] M. ASCE, AND E. J. CANDELL.[16]—Several contributors have taken the time to analyze this paper carefully, and have advanced conclusive thought and criticism regarding the experimental work. In response to these and to the many written and verbal comments made directly to the writers it is desired to state that the paper was not designed to advance a plea for a special method of sewage sludge disposal but rather to suggest a tool heretofore apparently not investigated and one which might find logical application in certain situations.

Mr. Abbott directs attention to the fact that by returning digested sludge to primary effluent for disposal the volatile suspended solids, as indicated by the writers' computations, may be increased from a total of 30% to a total of 65%. His position is well taken, but he fails to indicate that the 35% increase in volatile solids would result from the return to the primary effluent of those volatile solids in the digested sludge which are slow to digest and which, in general, are nonputrescible and have a low 5-day biochemical oxygen demand (B.O.D.). The volatile solids returned with the digested sludge are similar in composition to leaf mold, peat, or some such matter.

Mr. Abbott's reluctance to apply the 5-day B.O.D. test to sewage sludges is not unusual, although it is believed he would be on firmer ground if he were less reluctant to apply the test to digested sludge than to crude sludge. Results of the test as applied to sludges are erratic at times; but, for what the results may be worth, they are likely to measure, indeed, overmeasure, the oxygen demand placed upon a flowing stream, a lake, or salt water disposal site by digested sludge which will not strand. Mr. Abbott appears to be assuming that the remaining relatively stable volatile matter in the digested sludge will form sludge banks. The writers' admonition as stated in the paper,

NOTE.—This paper by A M Rawn and E. J. Candell was published in November, 1948, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1949, by W. E. Abbott, and Robert E. Leaver; and September, 1949, by A. L. Genter.

[15] Chf. Engr. and Gen Mgr., Los Angeles County Sanitation Dists., Los Angeles, Calif.

[16] Denver, Colo.

is not to return the digested sludge to primary effluent for disposal if there is danger of serious stranding.

Mr. Leaver questions the desirability of removing crude sludge from the sewage and then returning it in digested form for disposal. His criticism appears to stem from the fact that digested sludge is a valuable natural resource and should not be wasted to the degradation of streams. In the latter regard, Mr. Leaver directs attention to the accumulation of crude sludge in eddies in the rapidly moving stream waters of the State of Washington. Unfortunately, he tends to apply his reasoning to the Columbia River below Portland, Ore., into which river the crude sewage from cities on tributary streams and on the banks of the Columbia has been emptied for years without significant effect on the waters of that great stream. The effective reduction in the readily putrescible content of a large proportion of the suspended solids in the crude sewage of Portland can surely have only a beneficial effect on existing conditions; and, as applied to the Portland situation, the writers believe that the return of digested sludge to the primary effluent and its discharge into the Columbia River will escape detection by even the best of laboratory techniques.

Again the writers state that, if the solids will strand, the practice of returning digested sludge to primary effluent may not be satisfactory and probably should be practiced only experimentally and with caution until its merit is demonstrated. It is gratifying to note that the civil engineering department of the University of Washington, at Seattle, plans to extend the writers' research.

The writers agree with Mr. Leaver that digested sewage sludge is usually an excellent soil builder and that nothing should be allowed to curtail the activities of those who desire to reclaim and utilize this material for soil building —provided, of course, that the cost of such reclamation is borne by those who profit from the venture. Surely, nothing in the writers' discussion can be construed to indicate that they (who are among the most enthusiastic proponents of the return of digested sludge to the soil for agricultural betterment) desire to waste this material when it can be utilized properly and economically.

Mr. Genter questions the validity of the conclusions drawn from laboratory experiments on digested sludge. However, his position is not well taken with respect to laboratory digestion that is designed to illustrate the changes in sludge if it is allowed to continue digestion without further accretion of crude sludge. This condition is approached in the later stages of controlled stage digestion as currently practiced, and reasonably well demonstrated by the latter part of the curve of controlled laboratory digestion. Mr. Genter states that: "In the actual plant system fresh sludge is added daily to the digesters * * *." This statement is true only of the primary stage; it is not applicable to the later or ripening stages of modern digestion.

Mr. Genter's comments on the concentration of solids in the crude sludge do not apply to the conditions described in the paper. The solids content of the crude sewage is the controlling factor and not the concentration of the solids in the crude sludge. It appears logical that the greater the strength and solids content of the original sewage the less the primary effluent therefrom

will be affected percentage-wise by the return of digested sewage sludge which, having originally contained nearly 40% of the volatile suspended solids in the sewage, contains less than half that amount after some 30 days of digestion and only 10% of its originally measured 5-day B.O.D.

Mr. Genter's statement—"* * * the writer believes that this highly telescoped method of sewage and sludge disposal can serve only to postpone the day of real reckoning"—is not deemed appropriate by the authors. They feel that the recommendations contained in the paper will result in the final disposal of digested sludge as completely and adequately as if it were incinerated. Hundreds of thousands of tons of similar organic material from farms, forests, and meadows are disposed of annually in the same manner. Again, in response to Mr. Genter's criticism regarding the wasting of this valuable fertilizer, Mr. Rawn desires to state that the sewerage authority under his direction has produced and sold more digested sludge in the form of fertilizer than any authority west of Chicago, Ill. While still maintaining that the return of this organic material to the soil is of great value, he insists that there are locations where it cannot even be given away and where, if it is processed as a soil nutrient, the operation should be at the expense of those who will profit by the operation. Unless the manufacture of fertilizer is a necessary step in the operation of sludge disposal, at such locations the cost should not be borne by the public.

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DISCUSSIONS

STREAM FLOW VARIABILITY

Discussion

BY VEN TE CHOW, AND DANA M. WOOD

VEN TE CHOW.[21]—Following the computation procedure proposed in the paper, the variability index can be expressed by the formula:

$$I_v = \sqrt{\frac{\Sigma (y - \bar{y})^2}{n - 1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which y is the logarithm of the selected discharge at a 10% interval of the duration curve; $\bar{y}$ is the mean of y; $y - \bar{y}$ is the deviation from the mean; and n is the number of selected discharges, or $n = 10$ in the given procedure. In a practical operation, it is seldom convenient to take the deviations from the actual mean, since such deviations usually involve decimals which are cumbersome to handle when squared. For greatest convenience (especially in machine computation), another form of Eq. 1 is obtained by utilizing the mathematical principle that the sum of the squares of deviation from the mean is equal to the sum of the squares of deviation from zero, less the product of the total and the mean. Modified slightly, this principle is expressed

$$I_v = \sqrt{\frac{n}{n - 1} (\overline{y^2} - \bar{y}^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Eq. 2 is known as the Gaussian formula, which is used to estimate the standard deviation of population in statistics. In Eq. 2 $\overline{y^2}$ is the mean of the squares of y. The variability index can be defined mathematically and simply by this formula.

The numerical method of computing the variability index was primarily presented for the purpose of eliminating the effects introduced by variations in personal judgment. However, as the method used the values of discharges read from the duration curve as plotted, it did not eliminate the personal factor introduced in plotting the curve. A complete elimination of the influence of personal judgment is possible only by replacing the "eye fitting" by a mathematical process—that is, the values of $\bar{y}$ and $\overline{y^2}$ contained in Eq. 2 should be

NOTE.—This paper by E. W. Lane and Kai Lei was published in September, 1949, *Proceedings*.

[21] Hydr. Eng. Laboratory, Univ. of Illinois, Urbana, Ill.

computed directly from the raw data. Such data can be grouped by the method of statistical classification to reduce the number of data handled in computation. If necessary, the computed standard deviation can be corrected by applying the Sheppard correction.

Since the mathematical part of the paper has been omitted, the method of producing a duration curve from the given values of the variability index and the mean annual runoff cannot be checked readily. However, the stipulation (under the heading, "Use of the Variability Index") that—

> "* * * the ratio of the discharge exceeded 15.87% of the time to the discharge exceeded 50% of the time was equal to the variability index selected"

—would make the variability index greater than unity for most streams, because the discharge that is exceeded 15.87% of the time is usually greater than the discharge exceeded 50% of the time. Apparently this estimate does not agree with the values of the variability index listed in Table 1.

The knowledge of hydrologic statistics has been developed and extended greatly in recent years. Many investigators[17,22] have found that the distribution of most hydrologic phenomena is skew. Probability plottings of the Hazen type were found unsatisfactory for skew frequencies. The computation of asymmetric probabilities is now possible in the field of hydrology; and the development of a special probability paper[16,23] opened the way for a straight-line plotting of the duration curve. Therefore the study of stream flow would be advanced if a suitable asymmetric probability plotting could be adapted to it. The values of discharge should be plotted on a linear scale, so that the variability index would be equal to the standard deviation, in a statistical sense.

Many hydrologists like to use the coefficient of variation, C_v, instead of the standard deviation, σ, for the study of variability. The coefficient of variation is equal to the standard deviation divided by the mean, or

$$C_v = \frac{\sigma}{\bar{y}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

For purposes of comparison, the coefficient of variation is preferable to the standard deviation, because the former, being dimensionless, eliminates the effects of the difference from the mean. This elimination is desirable, especially for stream discharge in more arid regions where the annual rainfall tends to be smaller as well as more variable.

For any statistical study, the quantity of the available data is as important as the quality. As long as the quality is kept homogeneous, the longer record, or larger number of data, would make possible more accurate estimates of the frequencies of stream flow. W. D. Potter,[24] of the Soil Conservation Service,

[17] "An Asymmetric Probability Function," by J. J. Slade, Jr., *Transactions*, ASCE, Vol. 101, 1936, p. 35.

[22] "Floods Estimated by Probability Method," by E. J. Gumbel, *Engineering News-Record*, June 14, 1945, p. 97.

[16] "Straight Line Plotting of Skew Frequency Data," by R. D. Goodrich, *Transactions*, ASCE, Vol. 91, 1927, p. 1.

[23] "A Simple Method of Estimating Flood Frequency," by Ralph W. Powell, *Civil Engineering*, February, 1943, p. 105.

[24] "Simplification of the Gumbel Method for Computing Probability Curves," by W. D. Potter, *No. SCS-TP-78*, Soil Conservation Service, U.S.D.A., Washington, D. C., May, 1949.

United States Department of Agriculture, has made a probability study of rainfall and runoff. The study was confined to annual maximum values; but it is worthy to note that the frequency of distribution is a function of years of record, which is a measure of the number of data used. Since the variability index is a measure of the slope of the duration curve (a frequency curve), it should be some function of the length of record. Therefore, the variability index would be of more statistical value if the two were compared on the basis of the same length of record.

DANA M. WOOD,[25] M. ASCE.—The selection of an analytical method depends upon the use to which the results are to be put and the time available for the study. Cost is also a factor in any engineering office. Two suggestions concerning the duration curve may be of value to those desiring to extend this work.

(1) Instead of using a scale of "discharge in cubic feet per second per square mile," substituting "ratio to the average flow" for the period of record under investigation will result in relative variability curves of flow. The values of departures at 10% intervals can then be obtained similarly from this curve. This procedure eliminates the size of the drainage area and the magnitude of the runoff in any comparisons. A stream having a flow of 0.5 cu ft per sec per sq mile can be compared with one having a flow 2.5 cu ft per sec per sq mile with respect to the slopes of the duration curves—that is, the magnitude of the departures from the average, regardless of the size and other characteristics of the drainage area. The writer would like to know if this approach might in any way simplify the logarithmic calculations.

(2) The second point of importance is that the same period of years should be used when comparing different records in the same general locality. It is well known that the magnitude of the runoff varies in cycles of wet and dry years, although little enough is known about the amplitude and magnitude of the curve of variation. However, in the Tennessee basin and numerous other basins, the years from 1930 to 1948 have been generally in the "dry" part of a cycle. The previous nineteen years, from 1910 to 1929, were generally in a flat part of the cycle having normal annual variations. It is quite obvious that any analysis could lead to erronous conclusions if records in the two separate periods were compared. This point is well illustrated by the doubt regarding the analysis of records in different periods of years when attempting to determine the frequency of occurrence of floods of great magnitude.[26] However, for more distant localities, where the runoff cycles coincidentally may be radically different, further clarification and the development of methods of adjustment are needed.

This paper offers suggestions for many interesting exploratory studies, especially for the younger engineers training in these fields who are not afraid of undertaking many laborious calculations.

[25] Chf., Power Studies Branch, Div. of Water Control Planning, TVA, Knoxville, Tenn.

[26] "Possible and Probable Future Floods," by William P. Creager, *Civil Engineering*, November, 1939, p. 668.

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DISCUSSIONS

CONTROL OF THE HYDRAULIC JUMP BY SILLS

Discussion

BY MICHELE VIPARELLI, RALPH M. WEAVER, AND SHERMAN M. WOODWARD AND FLOYCE M. YATES

MICHELE VIPARELLI.[28]—In this remarkable study, the authors have devised a set of simple rules for determining the height, h, of a sill that will create a true hydraulic jump at a selected point in a rectangular channel, the flow being supercritical. Observations on two types of sill are reported—a sharp-crested weir and a vertical jog in the channel bottom. Of the two, the sharp-crested weir seems least practical because, more commonly, such a barrier would be in the form of a broad-crested weir. Convenient graphs are presented for the solution of equations developed from laws that govern the creation of the jump, and from laws that apply to fluid flow over weirs and in open channels. The use of nondimensional coordinates makes the application of the charts exceptionally simple.

The same problem of proportioning the sill can be solved very simply, without recourse to design graphs, with the following formula proposed by Guido Nebbia:[29,30,31,32]

$$\frac{H_2}{H_c} = 0.99 + 0.5\sqrt{\frac{\Delta H}{H_c}} - 0.025\,\frac{\Delta H}{H_c} \ldots\ldots\ldots\ldots (38)$$

in which $H_1 = d_1 + \frac{V^2_1}{2g}$; $H_2 = d_2 + \frac{V^2_2}{2g}$; $H_3 = d_3 + \frac{V^2_3}{2g}$; $\Delta H = H_1 - H_2$; and $H_c = \frac{3}{2}d_c = \frac{3}{2}\sqrt[3]{\frac{q^2}{g}}$. Eq. 38 can be applied directly for the case of the abrupt rise, in which the jog in the channel is generally created by excavation of the

NOTE.—This paper by John W. Forster and Raymond A. Skrinde was published in April, 1949, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1949, by En-Yun Hsu, and Steponas Kolupaila; and October, 1949, by Charles Jaeger, and Louis M. Laushey.

[28] Asst. of the Inst. of Hydraulics, Univ. of Naples, Naples, Italy.

[29] "Sul proporzionamento di traverse a soglia prolungata per dissipatori a risalto," by Guido Nebbia, *Acqua e Gas*, Roma, 1938.

[30] "Sui dissipatori a salto di Bidone, Basi teoriche," by Guido Nebbia, *L'Energia Elettrica*, Milano, 1940.

[31] "Sui dissipatori a salto di Bidone, Ricerca sperimentale," by Guido Nebbia, *ibid*, Milano, 1940.

[32] "Sui dissipatori a salto di Bidone Norme di proporzionamento," by Guido Nebbia, *ibid*, Milano 1940.

original bottom of a depth h between sections 0–0 and 3–3 (see Fig. 18). Then the designer generally knows the value of the unit flow q, the depth d_3, and the depth d_o which is dependent on the conditions upstream from section 0–0. Fig. 18 shows that:

$$H_1 = h + d_o + \frac{V^2_o}{2g} = h + H_o \quad \ldots\ldots\ldots\ldots\ldots\ldots (39a)$$

and

$$H_2 = h + d_3 + \frac{V^2_3}{2g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39b)$$

As $\Delta H = H_1 - H_2$ and H_e are given implicitly, it is possible to obtain H_2 from Eq. 38 and, then, h from Eq. 39*b*.

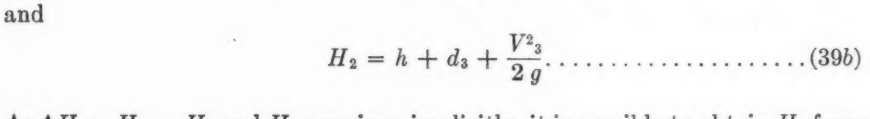

FIG. 18

In the case of a weir (sharp crested or broad crested) the solution for ΔH is by trial and error, solving for successively assumed values of h. Since the value of ΔH changes relatively little with corresponding values of h, the final solution generally requires no more than two trial computations.

Eq. 38 has an advantage over Fig. 6, which depends on sharp-crested weir coefficients, and is therefore convenient when the weir has a broader or other structurally resistant crest.

In their interpretation of the experimental results, the authors properly state that, had the length X been sufficient to develop the jump completely, the experiments would have checked the theory more exactly. Mr. Nebbia's experiments[29,31] confirm this observation. Under the heading, "Practical Application of Results," the authors propose "* * * that the curve for $X/d_2 = 5$ in Fig. 6 be used, under conditions of maximum discharge." Mr. Nebbia has presented evidence that there are conditions in which the highest value of h can be found at discharges less than the maximum, especially when the total head, H_1, of the flow in a lower stage is constant under varying discharges.[29,30,31,32]

RALPH M. WEAVER.[33]—In many instances it has been impossible to use a hydraulic jump because either the tailwater may have been too low, in which

[33] Hydr. Engr. I, Hydr. Laboratory, Norris, Tenn.

case the jump would not occur in the desired reach, or it may have been too high, causing the jump to be submerged. The authors have made an extensive study of two solutions for the first condition. In each, the water depth was raised, by weir action, to the depth required to form a jump.

A complete theoretical analysis, based on the assumption that submergence of the weir would not occur, was made for a weir and an abrupt rise in the channel floor. The formula for an unsuppressed weir with no velocity of approach was used with corrections applied to account for suppression of the weir, velocity approach, and nonaeration of the nappe. The theoretical analysis is interesting but, inasmuch as the values of the correction coefficients used in the derivation are approximate and the inflexibility of the formula limits its application, the practicality of the equation is questionable.

The results of the test agreed reasonably well with the theoretical analysis, but a discrepancy existed which may be explained by tests that were conducted by L. M. Laushey,[34] Jun. ASCE, R. M. Anderson,[35] J. K. Hallenburg, Jun. ASCE, and the writer[36] at Carnegie Institute of Technology, in Pittsburgh, Pa. These tests were conducted to determine the effect of a sill on the tailwater depth of the hydraulic jump. The curves in Fig. 6 indicate the presence of a velocity distribution such as that which existed in the foregoing tests. When the ratio of x/d_2 was small, the discrepancy between the theoretical and the actual curves was large, whereas the theoretical and actual curves were almost identical when x/d_2 was relatively large. At the larger values of x/d_2 the velocity distribution became more uniform than at any previous value of x/d_2. The theoretical formula applied actually only at the high values of x/d_2.

The tests[34,35,36] were conducted in a 40-ft flume 1 ft square with an adjustable inclined weir to permit fine adjustments of the tailwater depth. Sills $\frac{1}{8}$ in. wide, of varying heights extending the full width of the flume, were mounted in such a manner that the force of the water impinging on the sill could be measured by standard electrical strain gage methods. The initial depth was controlled by allowing the water to flow under an inlet plate that was fixed 0.2 ft above the floor of the flume. The general testing procedure consisted of measuring the force on the sill at various locations throughout the jump and the tailwater depths at the end of the jump. The results from the tests in which the end of the jump was over the sill were plotted as the dimensionless ratios of the initial depth, d_1, to the critical depth, d_c, and of the measured tailwater depth, d'_2, to the calculated theoretical tailwater depth, d_2, required to form a jump. It was found that the ratios of d'_2/d_2 ranged between 85% and 100% and that the values of d'_2/d_2 were a function of the ratio d_1/d_c and the sill height. The value d'_2/d_2 decreased as the sill height increased. As the sill height increased, a height was reached at which a jump would not form upstream from the sill.

It was also found that the force on the sill decreased rapidly to a minimum as the end of the jump was moved upstream to a position approximately over

[34] "Effects of Rectangular End Sills on the Hydraulic Jump, in a Level Rectangular Channel," by L. M. Laushey, Carnegie Inst. of Technology, Pittsburgh, Pa., 1947.

[35] "Forces on Rectangular End Sills in the Hydraulic Jump in a Level, Rectangular Channel," by R. M. Anderson, Carnegie Inst. of Technology, Pittsburgh, Pa., 1948.

[36] "A Study of Forces on Rectangular End Sills in a Hydraulic Jump in a Level, Rectangular Channel," by J. K. Hallenburg and R. M. Weaver, Carnegie Inst. of Technology, Pittsburgh, Pa., 1949.

the sill. The force then slowly increased to a constant value as the jump was moved further upstream. It was suspected that the change in force on the sill was due to a change in the velocity distribution from one end of the jump to the other as illustrated in Fig. 19. Observations of the action in a glass sided flume partly confirmed this theory.

The nonuniform velocity distribution increased the momentum above that existing for a uniform velocity distribution at the same tailwater depth. With the increase in momentum the pressure requirement was reduced, thereby decreasing the depth. This condition may be verified by substitution in the pressure-plus-momentum equation which must be satisfied.

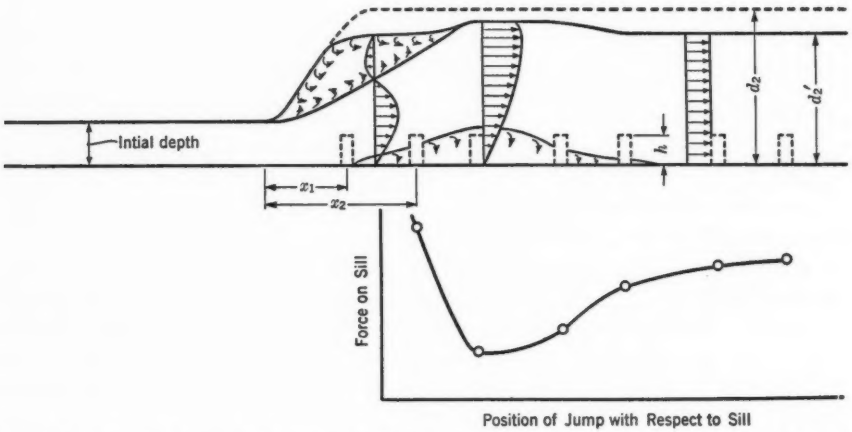

FIG. 19.—POSSIBLE VELOCITY DISTRIBUTION WITHIN A HYDRAULIC JUMP WITH A SILL AND THE FORCE ON THE SILL AT VARIOUS POSITIONS OF THE SILL

An analysis of the results showed that: (*a*) These tests proved that placing a sill at the end of a hydraulic jump reduced the tailwater depth below the theoretical depth required to form a jump; and (*b*) the reduction in tailwater depth was caused by a change in the velocity distribution due to the introduction of the sill into the jump. The results further disclosed that the assumption of uniform velocity such as was made by Messrs. Forster and Skrinde was not entirely justified.

The tests conducted at Carnegie Institute of Technology[34,35,36] were performed under conditions similar to the tests made at the State University of Iowa in Iowa City, as reported in this paper, with one exception. In the former an attempt was made to maintain the downstream depth and the tailwater of the jump at the same level instead of using the sill to raise the tailwater. This attempt was not always successful, but it was never so unsuccessful that the sill became the control. The control was located well downstream from the sill at all times.

In view of the fact that a nonuniform velocity distribution existed in the tests conducted at the State University of Iowa, a reduction in the jump tailwater should occur. The theoretical curve should have been determined by

the expected tailwater and not by the tailwater computed by the hydraulic jump formula. The tailwater that should occur would be less than that computed. Substitution of the expected tailwater into the theoretical formula will reduce the value of h/d_1, thereby lowering the curve nearer to the actual results.

SHERMAN M. WOODWARD,[37] HON. M. ASCE, AND FLOYCE M. YATES.[38]—In checking through this paper the writers discover at once that it presents an interesting mathematical exercise. They are of the opinion that every studious, serious-minded engineer, before applying the results contained in the paper to actual practice, would feel it necessary to verify the rather unusual and difficult algebraic transformations involved in the preparation of the paper. The usual engineer in active practice, however proficient he may have been in mathematics while in college, rapidly forgets many details, and soon loses the facility of the professional mathematician. It is regretful, therefore, that the authors of the paper did not include a few more brief descriptions of the steps followed in its mathematical treatment.

As an attempt to remedy this deficiency, the writers venture to suggest the following explanatory notes, particularly to the young inexperienced engineer who contemplates contributing further original studies in this field. It should be remembered that there are, of course, many possible variations in details.

If the value of V_2 from Eq. 3 is substituted in Eq. 2, the resulting equation is a cubic in d_2, which means that d_2 has three values in terms of the other variables. It is also rather obvious that $d_1 - d_2$ is a factor of both members of the equation. Dividing both sides by $d_1 - d_2$ reduces the equation to a quadratic, which may be easily solved by any of the usual methods. It must be remembered that in eliminating the factor $d_1 - d_2$ one of the roots of the equation—namely, $d_2 = d_1$, is discarded.

Substituting $g = \frac{\gamma}{\rho}$ the quadratic becomes

$$d^2_2 + d_2 d_1 = \frac{2 q V_1}{g} \ldots\ldots\ldots\ldots\ldots\ldots (40)$$

By the use of Eq. 3 the second member of this equation may also be written as $\frac{2 d_1 V^2_1}{g}$ or $\frac{2 q^2}{g d_1}$. Of these three possible forms the best one to use in the particular circumstances should be determined by the existing situation. The authors apparently chose the third or last of the three forms for their subsequent derivation of Eq. 13*a*. This is a natural procedure because Eq. 40 expresses the mathematical relation necessary for the formation of the jump, and in it the value of q given by Eq. 11 may be substituted. This substitution introduces the height necessary for the assumed controlling sharp-crested weir and so relates the effect of the weir to the formation of the jump.

In following this procedure, however, formidable and somewhat bewildering complications arise. In squaring both sides of Eq. 11, since C_2 and

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[38] Eng. Aide, TVA, Knoxville, Tenn.

C_3 as expressed by Eqs. 12*a* and 12*b* are both binomials, their squares are each trinomials and their product contains nine terms corresponding to the nine terms contained within the large bracket in Eq. 13*a*. These terms contain powers and products of the two ratios in Eqs. 12*a* and 12*b*. It is essential sooner or later to eliminate all redundant forms of the variables. Having decided to retain as variables N and r as defined in Eqs. 13*b* and 13*c*, Fig. 1(*b*) shows that $H = d_2 - h$. Therefore, $\frac{H}{d_2} = \frac{d_2 - h}{d_2} = \frac{N - r}{N}$ and $\frac{h}{H} = \frac{r}{N - r}$.

These substitutions result in Eq. 13*a* as printed with slight variation in the third significant places. To reduce Eq. 13*a* to a standard form it is necessary to use the binomial theorem to expand the indicated powers, to perform the indicated multiplications, and to remove the denominators. The expansion produces nearly fifty terms, but by combining similar terms the number is reduced to eight. The equation then becomes

$$N^6 + N^5 = -0.113\,r^7 + 0.830\,r^6\,N - 3.019\,r^5\,N^2 + 6.741\,r^4\,N^3 - 9.804\,r^3\,N^4 + 9.206\,r^2\,N^5 - 5.069\,r\,N^6 + 1.228\,N^7 \ldots\ldots (41)$$

In plotting curve (*a*) in Fig. 2 to represent Eq. 41, it is necessary to compute a few points by the method of successive approximations. Eq. 41 is of the seventh degree in both N and r, which means that if a convenient integral value for either variable is substituted in Eq. 41 there may be seven corresponding values of the other variable which will satisfy the equation. Some of these values, perhaps most of them, may be imaginary. Fortunately the curve appears to be rather flat so that a small number of points will suffice to determine it.

It is convenient to begin with points near the origin where the values of the coordinates will be relatively small. Thus, assuming that r is to be the independent variable and N the dependent variable, let $r = 0$. Then Eq. 41 reduces to $N^6 + N^5 = 1.228\,N^7$ or $1.228\,N^2 - N = 1$.

Solving the quadratic, $N = 1.396$ or -0.583. Only positive values are probably of practical interest.

To continue, let $r = 1$ in Eq. 41. Applying Descartes' rule of signs to the resulting seventh-degree equation indicates that there may be seven positive roots or values of N which satisfy the equation. By successive trial it is found that $N = 3.56$ is one such value. Continuing similarly, sufficient points may be found to complete the drawing of curve (*a*) in Fig. 2.

It is the writers' aim to emphasize the importance of making clear the complex mathematical processes used, so that the practicing engineer may utilize with confidence the curves shown in Figs. 6 and 7.

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DISCUSSIONS

CONTINUOUS ARCHES AND BENTS ANALYZED BY COLUMN ANALOGY

Discussion

BY POWELL P. W. TSU

POWELL P. W. TSU.[23]—The analysis of an indeterminate structure is essentially a study of the relation between the elastic deformations of each member comprising it, and the selection of sections that will maintain structural continuity. In the case of plane structures, these elastic deformations occur in response to moment, axial thrust, and shear. Except where the member is short and deep, the elastic deformations resulting from axial thrust and shear are generally small and negligible as compared to those produced by bending moment. Following this line of reasoning, the column analogy, as developed by Hardy Cross, Hon. M. ASCE, results in a simple formula for continuity moments, in which the deformations from axial thrust and shear are neglected. The method is useful in the solution of a three-redundant structure because it is susceptible to a systematic computation when arranged in tabular form.

The extension of the column analogy to the analysis of continuous structures, as presented by Mr. Yu, is a contribution to engineering literature despite the fact that its range of application is limited to certain cases, the same as many other methods and modifications in the field of structural analysis.

In its application to the analysis of arch spans (particularly when the arch is flat), the column analogy is good only for preliminary investigations because it neglects the effect of axial deformation, an omission that seems unwarranted. In other words, unless axial deformation can be taken into consideration, it is questionable if the author's conclusion (that his method is especially useful as a device for reviewing designed structures) is justified in the case of arch spans.

In Example 2, the author offers a continuous bent for analysis. The same problem was solved by the writer using the method of moment distribution. With the aid of an ordinary slide rule, it required only 8 hours to finish the entire computation, including the values of influence ordinates for moment,

NOTE.—This paper by T. M. Yu was published in February, 1949, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1949, by Jack R. Benjamin; and October, 1949, by Leslie J. Meszaros, A. A. Eremin, and J. J. Polivka.

[23] Design Engr., U. S. Naval Port Facilities, Shanghai, China.

thrust, and shear, corresponding to the data in Fig. 8. Although a comparison with the author's working time is of no great consequence (the speed of computation depending to a large measure on the degree of acquaintance with each particular method), the relative accuracy of results is a proper criterion. Since presentation of other methods is beyond the scope of discussion, the writer's computations cannot be presented; but the results show, invariably, a smooth curve in influence ordinates for each stress function whereas the author's results show marked divergence in the remotest spans. The errors are most likely mistakes in computation because systematic errors in computation would produce influence ordinates that revealed a general disagreement, rather than the isolated ones actually produced. It is clear that the column analogy is limited as a basis for analysis because it offers no method of applying an independent check. Mistakes in computation will be carried along and thrown into the final result unless (as demonstrated by the author) they can be discovered by the equality of reciprocal deflections, which is a check not available until the computation is completed. For the case of a long series of continuous spans, the successive combination of spans necessary to arrive at a single span wherein the particular stress functions are to be studied would entail so many tabulations that a slight mistake in computation, if not rigidly checked at the end of each process of combination, may result in a final, cumulative, gross error.

Furthermore, in getting influence lines by using the Mueller-Breslau principle, the computation of deflection ordinates in remote spans, due to unit strains corresponding to the desired influence functions, permits a rather high degree of precision as shown by the significant data in Fig. 7. In the writer's solution of the same problem, a uniform degree of precision was controlled by using an ordinary slide rule. In tracing influence lines for stress functions at sections other than those illustrated by the author, it is necessary to find as many independent sets of indeterminate moments by column analogy (moments due to applied unit strains at a particular section) as there are numbers of sections intersected, and then to compute the deflections. By moment distribution, it is only a matter of simple arithmetic applied to the principle of statics, to compute influence ordinates at any number of sections, using only the influence lines for fixed-end moments and the joint moments distributed from a joint with an arbitrary unbalanced moment.

The column analogy computation gives the combined effects of sidesway and joint rotation but it affords no clue as to the proportions between them. Moment distribution yields that result directly and is much preferred.

Example 1 in the paper shows clearly the merit of Mr. Yu's method. The directness with which the substitute members are found simplifies the solution. The same problem, if solved by the method of moment distribution, would require two sidesway corrections and two simultaneous equations, making it perhaps less favorable than the author's method.

To summarize, the writer feels that, although the author's contribution to engineering literature is highly commendable, its field of application in the analysis of continuous structures is limited, being neither as rapid nor as accurate as many other methods.

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DISCUSSIONS

WIND STRESSES IN SEMI-RIGID CONNECTIONS OF STEEL FRAMEWORK

Discussion

BY GEORGE S. MURPHY

GEORGE S. MURPHY,[15] ASSOC. M. ASCE.—The determination of the actual strength of semi-rigid connections depends (as the author states) on the availability of experimentally determined empirical curves, one for each joint combination, assuming identical quality of workmanship amd materials in every case. Therefore, the problem is to discover what limiting assumptions will keep one on the safe side of uncertainty.

Seldom does ordinary design procedure go beyond computations for the tension, bearing, and shear in the rivets or welds, while assuming the base material of the members and the connection pieces to be fully nondeformable—despite a full awareness, by the designer, of the flexibility and plasticity of the parts of a standard connection. The frame (excluding wind bents) is designed assuming zero end moments (that is, assuming that there is no restraint against the pulling away of the top of a connection); and yet, subsequently, the superimposed concrete floor is designed as if the framing connections were fully continuous. Running cracks in the concrete floor, along the steel girders to which the steel floor beams are framed, are not unknown occurrences. They bear witness to the ill effects of this contrasting approach to the design of different parts of the same structure; and any effort to correct this approach should be encouraged.

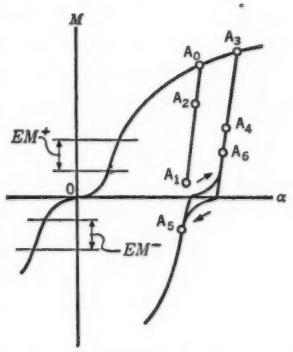

FIG. 13.—STRESS DIAGRAM FOR CONNECTION AT A (FIG. 2), SHOWING POSSIBLE EFFECT ON CURVE FROM LOOSENESS AT TIME OF REVERSAL OF MOMENT

The general type of stress diagram for connections (Figs. 3 and 6), in which moments are plotted against angular deflections, might be questioned in the area of positive or negative increase from zero moment, as demonstrated by

NOTE.—This paper by Basil Sourochnikoff was published in February, 1949, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1949, by C. Batho.

[15] With Samborn, Steketee, Associates, Toledo, Ohio.

Fig. 13. An analogous illustration would be the free play in the steering column of a car or boat when the direction of the turning force is reversed. A break in the elastic curve, such as this, would have a considerable effect in altering the distribution of moments in the structure where not all the joints would be subjected to a reversal of moment. Stress states A6 and A4 would then not be identical.

In defining the conditions for keeping a connection within the elastic range, the author's association of algebraic and absolute numbers may lead to confusion. This danger is particularly apparent when one considers that a given connection may have different elastic limits ($E\ M$, Fig. 13) for different directions of rotation. Eq. 12*a* would be more mathematically precise in the form:

$$M_{B5} > -|E\ M_B|_{\text{neg.}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

In other words, M_{B5} is algebraically greater than the negative of the absolute value of the elastic limit of the joint when rotated in the negative direction.

Conclusions (1) and (2) are undeniably correct. Conclusion (3), however, is a qualitative conclusion within more or less indeterminate limits; and caution is required to avoid its improper application to specific cases. Judgment is required in estimating the capacity inherent in the connection of undergoing "large" inelastic deformations without failure, and in estimating the magnitude of the inelastic deformations it will be required to undergo. Furthermore, the statement that "adequacy" of the connection for combined gravity and wind loads will be found if "adequacy" to resist wind moments alone is found, when the latter are computed assuming elasticity of the connections, is contingent, not only upon sufficient deformability of the connection, but also upon a sufficiently large ratio of the wind moments to those caused by the gravity loads. Otherwise, in windless climes, no connections would be required. This is recognized by the design specification that, in any case, the connection be "adequate" for the gravity loads.

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DISCUSSIONS

EFFECT OF DRIVING PILES INTO SOFT CLAY

Discussion

BY W. S. HOUSEL

W. S. HOUSEL,[42] M. ASCE.—An investigation, similar to that reported in this paper, was conducted at the Union Station in Detroit, Mich., in connection with the design of grade separation structures on the John C. Lodge Expressway.[24] It is pertinent to compare the conclusions drawn in that case with those presented by the authors. In some respects the later investigation confirms the authors' deductions and provides some additional data regarding this problem about which experimental results are still rather fragmentary. The primary objective of the tests was to study methods of controlling the volume displacement due to pile driving which sometimes has harmful effects on adjacent structures or utilities. With respect to this phase of the investigation it was found that the use of tapered, open end steel piles of the type used did eliminate a major part of the volume displacement. Subsequent loading tests indicated that those piles in which the clay core was not removed had an ultimate carrying capacity as controlled by the soil comparable to that measured on the piles filled with concrete.

Largely as a result of the publication of the earlier investigation made by the authors it was decided to conduct some parallel studies in connection with the Union Station pile tests to study remolding and moisture content changes due to pile driving. The tentative conclusions drawn from the investigation, and which may be compared with the authors' deductions, are as follows:

1. The Detroit clay deposit consists of at least two depositions of lacustrine clay showing a different reaction to pile driving in so far as loss of shearing resistance is concerned. The physical characteristics of these deposits, as determined by index tests, show some similarity to those reported by the authors.

NOTE.—This paper by A. E. Cummings, G. O. Kerkhoff, and R. B. Peck was published in December, 1948, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1949, by Leonardo Zeevaert, Jacob Feld, Joseph C. Fegan, and Gregory P. Tschebotarioff; June, 1949, by Philip C. Rutledge, A. W. Skempton, James R. Schuyler, and B. A. Kantey; September, 1949, by D. P. Krynine, and Robert F. Leggett; October, 1949, by Stuart B. Avery, Jr., and Stanley D. Wilson, and George L. Freeman and James D. Parsons; and November, 1949, by L. LeRoy Crandall.

[42] Associate Prof., Civ. Eng., Univ. of Michigan, Ann Arbor, Mich.

[24] "Investigation to Determine the Driving Characteristics of Piles in Soft Clay," by W. S. Housel and J. R. Burkey, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng., Rotterdam, 1948, Vol. V, pp. 146–154.

2. Above approximate El. 530 at the Union Station, clay within 3 in. of the pile suffered loss in shearing resistance comparable to the value when remolded. Determination of recovery was based on two types of shear tests made approximately 2.5 months after pile driving. Below El. 530 there was no apparent loss in strength due to pile driving at a distance of approximately 3 in. from the pile. No tests were made to determine recovery of the soil with time. Shear values in the lower clay deposits 2.5 months after driving were substantially higher than the original shear strength found in the check boring but within the range of variation found in a series of exploratory borings. These results were confirmed by two similar borings at a distance of 3 in. from the piles.

3. In two borings at a distance of several feet from the piles there was no evidence of loss in shearing resistance due to remolding in either the upper or the lower clay deposit. In one case shear values in the lower deposit were greater than in the check boring but within the range found in the series of exploratory borings. This would tend to confirm that the higher shear values found in the lower clay were the result of a natural variation in the deposit rather than an alteration produced by pile driving.

4. There was no positive evidence of any change in moisture content in either the top or bottom clay deposit but at one depth where there was a particularly soft and highly plastic clay there were scattered results from which it might be concluded that a decrease in moisture content had taken place within 3 in. of the piles.

In presenting these data the writer wishes to point out that, although the results reported are based on duplicate borings and duplicate laboratory shear tests and while they indicate some trends, there is such a wide variation in shearing resistance and physical characteristics in the natural state that the statistical basis of comparison becomes extremely narrow, making the available data comparatively fragmentary. For example, in both investigations there are only data at two points with respect to distance from the piles, one at approximately 3 in. and one at several feet, giving little measure of the boundary of zones within which remolding is important. Perhaps more important is the possible variation with time, particularly in the rate of recovery of the loss in strength. On this point there is quite incomplete coverage in the results reported from both investigations, and considering the character of the tests and the expense involved it is not likely that such information will be made available under ordinary circumstances.

It would appear to the writer that there are other, and perhaps more important, aspects of the effect of driving piles in soft clay to which attention might be directed and on which observation and experience may provide more conclusive information. For one thing, it is a matter of common experience in pile driving that the load-carrying capacity of the piles as determined by conventional pile driving formulas may be very small at the time the driving is completed, but within a period of hours or a few days friction piles "set up" and offer much greater resistance to penetration. In fact, it is common practice to take advantage of this well-known phenomenon in meeting specifications that require a certain pile capacity as determined by commonly used formulas.

Some years ago the Detroit Edison Company conducted a rather extensive series of load tests on friction piles to determine the recovery of the soft clay from the detrimental effects of pile driving. These tests were made by direct loading on two 16-in. (outside diameter) steel piles, approximately 65 ft long, which were driven at the company's Delray Power Plant in Detroit. Soil conditions in the area are comparable to those under discussion by the authors, being in the River Rouge area of Detroit where the entire depth of clay to hardpan consists of a soft-plastic lacustrine deposit.

Immediately after the piles were driven, load testing equipment was set up and direct loading tests were conducted at intervals starting from about 2 hours after the piles were driven and continuing for about 60 days. The results of these tests have never been published but seem to be pertinent to the subject now under discussion and do provide evidence of the nature of the temporary loss in strength and subsequent recovery of such clays after pile driving. Observations of these phenomena indicate a rapid recovery in shearing resistance accompanied by parallel changes in elastic properties which are seldom associated with changes in moisture content and consolidation of the soil mass.

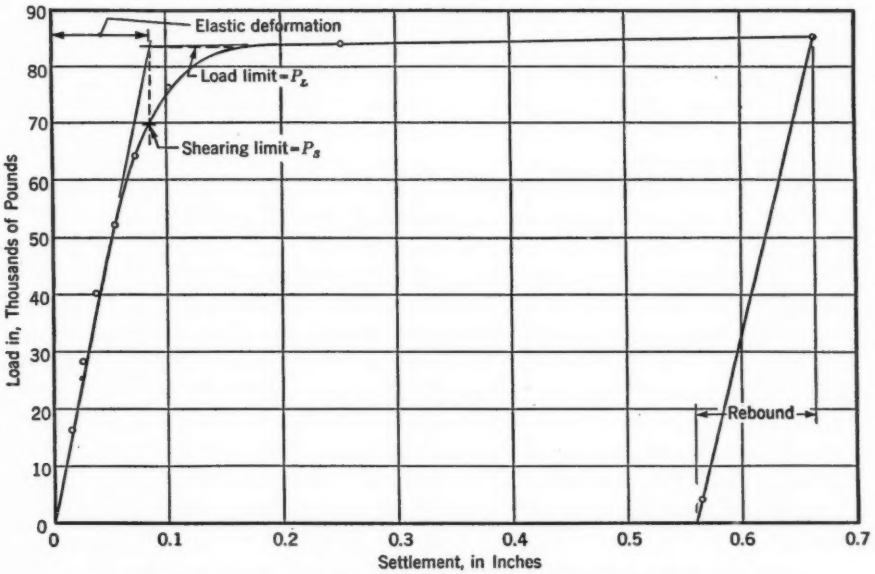

FIG. 35.—LOAD SETTLEMENT DIAGRAM, 16-IN. STEEL CYLINDER IN SOFT BLUE CLAY; TEST NO. 20, DELRAY POWERHOUSE, DETROIT, MICH.

A typical load settlement diagram taken from the series of thirty-six tests conducted on the two piles is shown in Fig. 35. This diagram indicates the resistance characteristics of the clay mass measured by loading a so-called friction pile. Even though the clay is soft or highly plastic it gives evidence of an extended range of elastic behavior shown by the linear relation between load and settlement extending close to the load limit. After the load limit has

been exceeded there is a stage of rapid progressive settlement shown by an almost horizontal load settlement line followed by a rebound or elastic recovery after the release of the load. Throughout the series of tests it was found that the rebound was approximately equal to the total elastic deformation as determined by the intersection of the elastic load settlement line with the failure line, a point which is designated as the load limit, P_L. The shearing limit, P_S, has been selected as the load carried at a deformation equal to the total elastic deformation. The difference between the shearing limit and the load limit in the curved portion of the load settlement diagram is considered to be the point resistance. In this connection it should be pointed out that the friction pile in Fig. 35 was tested with the clay core inside the pipe extending almost to the top of the pile, a condition resulting immediately after driving. In Fig. 36 is shown a comparison between rebound and total elastic deformation for the series of tests on such a friction pile, supporting the conclusion that the rebound and total elastic deformation were equal.

In Fig. 37 is shown a load settlement diagram for the same pile after the clay core had been removed for the purpose of completely eliminating point resistance. In the first place it is evident that the difference between the shearing limit and the load limit has almost disappeared and the failure of the pile is quite abrupt. Of greater importance, however, is the comparison afforded by immediate repetitions of load shown in Fig. 37. This type of test was made a number of times throughout the series of tests demonstrating beyond any reasonable doubt that friction piles in such cohesive clays could be loaded to rapid progressive settlement without impairing the shearing resistance of the material.

The variation in load-carrying capacity of the pile, which may be assumed to be equivalent to the shearing resistance of the soil, provides the most important data available from these tests. This variation is shown in Fig. 38 in which the shear limit as previously defined has been plotted for one of the piles with the time in days since driving as the abscissa. A few hours after driving the shear limit was approximately 25,000 lb. This increased rapidly and in a period of 14 days the shear limit had increased to approximately 77,000 lb, or about three times its initial value. At this point the pile had developed a capacity approximately equal to that which had been estimated from the independent shearing resistance tests in which the average shear for this depth was approximately 275 lb per sq ft. A duplicate test on the second pile at this time provided a very satisfactory confirmation of the total shearing capacity.

One of the most interesting observations which this series of tests provided resulted from what might be termed an accident that occurred after the first 14 days had elapsed. Each of the preceding piles was tested to rapid progressive failure and at this point it became necessary to put an additional length of pipe on this pile in order to continue the tests. The construction superintendent who had been assisting on the project was very much interested and very anxious to do everything possible to bring the investigation to a successful conclusion. Consequently, in order to eliminate any possible settle-

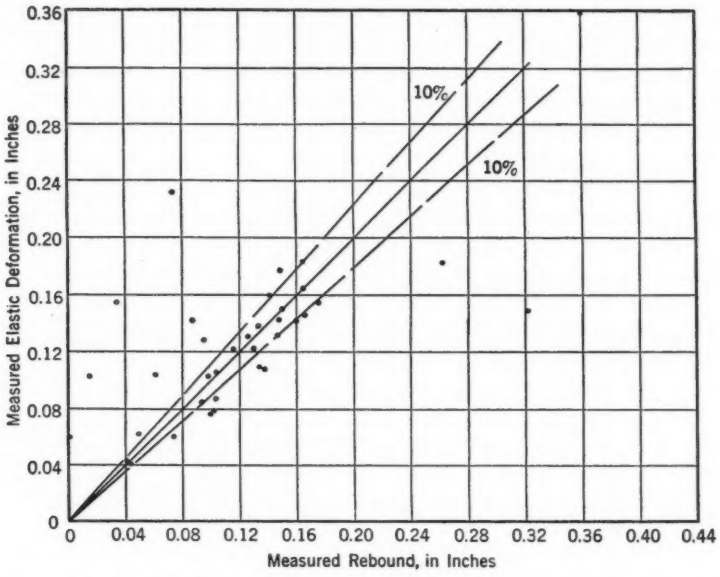

FIG. 36.—COMPARISON OF ELASTIC DEFORMATION AND REBOUND, 16-IN. STEEL CYLINDER IN SOFT BLUE CLAY, DELRAY POWERHOUSE, DETROIT, MICH.

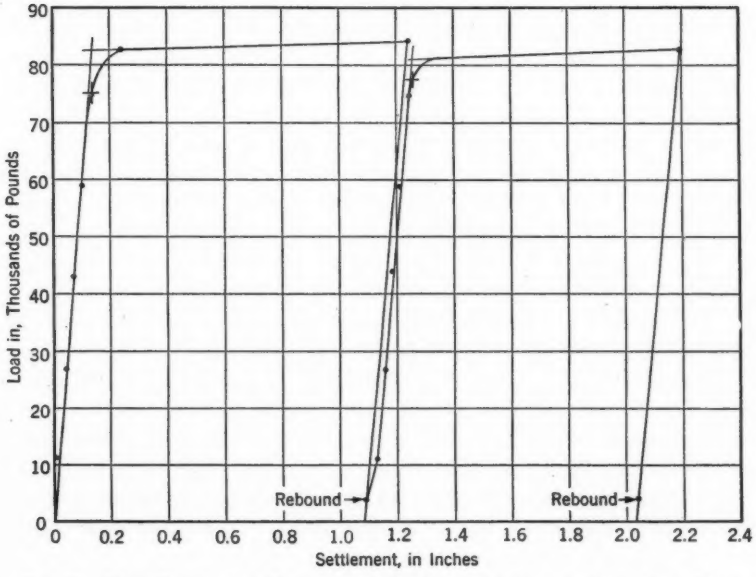

FIG. 37.—LOAD SETTLEMENT DIAGRAM, 16-IN. STEEL CYLINDER IN SOFT BLUE CLAY; TEST NO. 32, DELRAY POWERHOUSE, DETROIT, MICH.

ment from incomplete seating of the new length of pipe at the joint, he brought the pile driver back on the job and drove the pipe about 3 in., possibly striking no more than six blows. The surprising result is shown on the curve in Fig. 38. The next test after redriving of the pile shows an abrupt decrease in shearing resistance which was substantial although not as great as the original loss in strength.

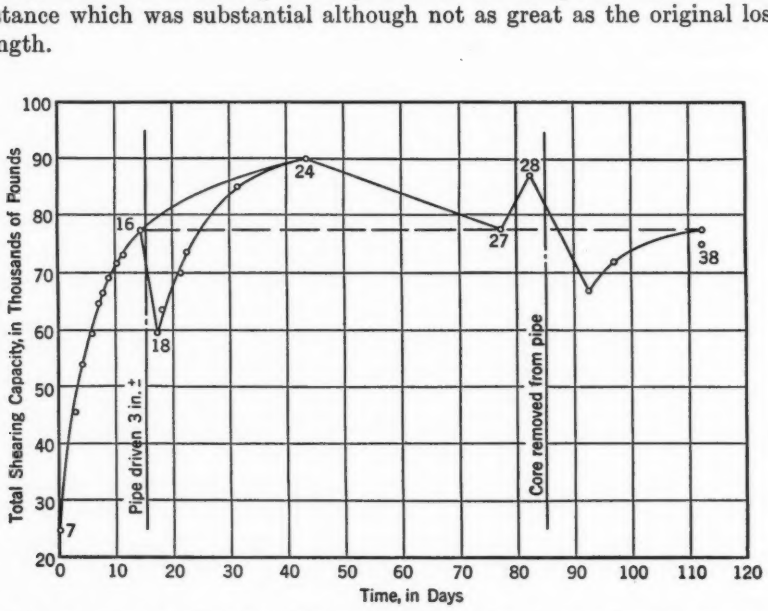

FIG. 38.—VARIATION IN TOTAL SHEARING CAPACITY WITH TIME ELAPSED AFTER DRIVING 16-IN. STEEL PIPE (FRICTION LENGTH, 65.75 FT TO 67.86 FT); DELRAY POWERHOUSE, DETROIT, MICH.

It seems obvious that the slight movement of the pile involved in redriving could not have produced any significant amount of remolding of the soil or change in the moisture content. It should also be pointed out that the displacement involved was only a little more than that which took place in each of the static load tests in which the soil was carried to rapid progressive displacement. It may be stated with considerable certainty that deforming the soil (by rapid shearing displacement under static load) did not impair its strength; but the shock of just a few blows of the hammer produced a marked decrease in strength. Furthermore, the decrease in shearing resistance was accompanied by a parallel decrease in elastic deformation as recorded in both the rebound and the total elastic deformation. These data are shown in Fig. 39 and, in the writer's opinion, present very strong evidence that the primary source of the loss in strength and its subsequent recovery can be traced back to the very source of cohesion in adsorption or molecular orientation of soil moisture. The phenomena may be called a thixotropic effect, which is characteristic of colloidal systems and has been observed in clays of the character under discussion.

This series of tests produced evidence on at least one other interesting phenomenon. It may be mentioned in passing even though its basis is some-

what speculative. As the tests were conducted the shearing resistance increased with time up to a maximum of 43 days as shown by test No. 24 (Fig. 38) where the shearing limit is approximately 90,000 lb. This total capacity was substantially greater than had been anticipated from previous shearing resistance tests and greater than the shearing resistance on the duplicate pile that was being tested. At the same time the test data gave evidence of a

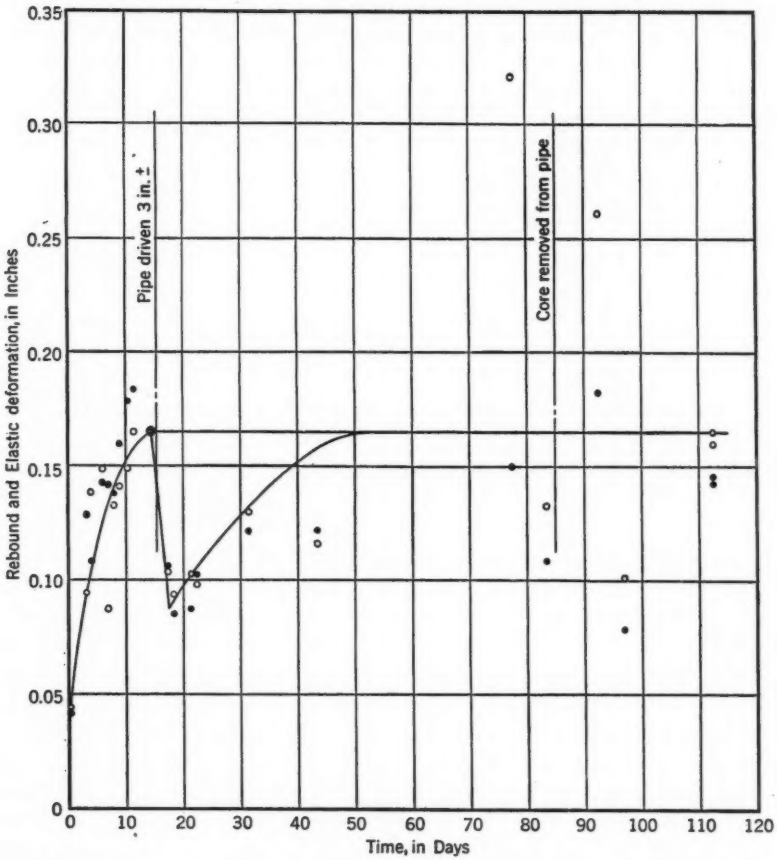

FIG. 39.—VARIATION IN ELASTIC DEFORMATION AND REBOUND WITH TIME ELAPSED AFTER DRIVING, 16-IN. STEEL PIPE (FRICTION LENGTH 65.75 FT TO 67.86 FT); DELRAY POWERHOUSE, DETROIT, MICH.

disparity between elastic deformation and shearing resistance which led to the speculation that a dynamic point resistance was being built up as a result of repeated tests which produced rapid progressive penetration of the pile. The magnitude of this dynamic point resistance was quite high, approaching the passive pressures that might be anticipated under the static head of some 65 ft.

It was then decided to do two things to obtain further evidence on this phenomenon. In the first place, the pile was left without load repetition for

approximately 30 days and then tested as shown in test No. 27 (Fig. 38). The total shearing capacity at that time had dropped back to approximately 77,000 lb. This was fairly close to that which had been anticipated from other test data, indicating that the high intensity of pressure developed at the point of the pile had been dissipated. In the succeeding test No. 28, however, this dynamic resistance was substantially recovered. In the next test the core was removed from the pipe in order to eliminate point resistance completely. This removal was accompanied by considerable disturbance caused by the inflow of soil into the bottom of the pipe, as it was found in the soft clay that either a head of water or a substantial core was required to prevent such loss of ground. In the subsequent tests, as shown in Fig. 38, the shearing capacity of the pile was recovered and the series of tests was terminated.

Acknowledgment.—The tests at the Union Station, referred to in this discussion, were conducted jointly by the Michigan State Highway Department, the Wayne County Road Commission, and the Union Metal Manufacturing Company.

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DISCUSSIONS

MULTIPLE-PURPOSE RESERVOIRS
A SYMPOSIUM

Discussion

BY DANA M. WOOD, AND CLARENCE S. JARVIS

DANA M. WOOD,[30] M. ASCE.—Considering the papers of this Symposium as a whole, the reader is immediatley impressed by the number and authority of agencies having an interest in federal multiple-purpose development in particular, and a nearly equal interest in private developments. Assuming, for example, that the Corps of Engineers, United States Department of the Army, under Congressional authorization, has a paramount interest in planning and design, because of flood control or navigation functions or both, it must: (1) Obtain the approval of the Federal Power Commission on the design and the comprehensiveness of the river development scheme; (2) cooperate with the Bureau of Reclamation if irrigation is involved or arrange for the United States Department of the Interior to sell any power available; (3) perhaps coordinate power planning with private utilities if an independent market is not available; (4) secure recommendations from the Fish and Wildlife Service of the Department of the Interior; (5) determine the extent of participation of the National Park Service of the same department with respect to recreational possibilities; (6) procure the hydrometeorological services of the United States Weather Bureau and the United States Geological Survey; (7) ascertain public health requirements; and then (8) receive cooperation and assistance at lower state and municipal levels. Historic and archeological investigations must be included by the National Park Service, assisted by the Smithsonian Institution in Washington, D. C., and perhaps by local universities. The soil conservationists have an interest. All this requires special appropriations for each agency, or a shifting of funds from one to the other, necessitating government budget requests and reviews, or special agreements. How complicated the life of the government planning and design engineer is becoming and how divided the authority!

NOTE.—This Symposium was published in March, 1949, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1949, by Herman A. Hilton, Wesley E. Gilbertson, Roger E. Amidon, L. C. Bishop, and A. H. Davison; and September, 1949, by John S. Cotton, and Abraham Streiff.

[30] Chf., Power Studies Branch, Div. of Water Control Planning, TVA, Knoxville, Tenn.

Little has been said about the necessity of obtaining local public approval, coordinating, and compromising the appeals of special pressure groups, or overcoming selfish and individual attitudes at all levels. Of course, the greatest good to the largest number of individuals is supposed to be the controlling theory, but political and financial controls can violate this. One of the greatest evils of today is an unwillingness to compromise issues on any level from the individual to even the international. The question is too apt to be raised, "How does this affect the individual, or the community, or the State?"

Yes, a national water policy is needed, but how is one to be obtained? Is Congress ready to modify its stipulations that one agency sell its power at the switchboard while another builds its transmission lines to the market, or is any particular agency willing to depart from its own pet theory of allocation of costs of multiple-purpose objectives in determining project economics? Is flood control of greater or less importance than power except for reasons of political expediency? Are local benefits sufficiently large to outweigh local taxes on government projects? Should theoretical rather than actual interest be charged against a government project which competes with private business? Will nonreimbursable appropriations be made at the expense of the United States at large so that purchasers of water and power will not be required to pay for services provided for the general public in the same restricted area? Shall large landowners receive the same benefits from irrigation as small landowners? Shall residents within a flood control district contribute to the project on some equitable basis or shall the federal government bear the entire expense? Shall governmental planning be allocated to numerous bureaus or be consolidated in a new bureau? One can go on indefinitely in raising questions requiring compromise which probably never will be actually compromised until the character of the individual and the groups is improved and selfish interest is largely eliminated.

A national water policy must be flexible because no two cases are exactly alike. In one case, it may be necessary to sell power at the switchboard to the one available utility, thereby continuing a monopoly, because any other market could not be built up readily. In another case, the public interest might be hurt by following such a policy. The residents within a proposed flood control district might not have the financial ability to participate, no matter how desirable the flood protection might be, because of already existing financial responsibilities. Many municipalities are overburdened with heavy debt, perhaps as a result of mistaken past policies, and cannot undertake badly needed improvements. This condition, of course, raises the important question as to whether a community which has been thrifty by maintaining a pay-as-you-go basis should now be penalized to assist through taxes a less thoughtful community. Water laws and practices in one section of the United States differ from those in another. Basic fundamental questions are woven all through the problem of a national water policy.

Desirable as some national water policy may be, it seems likely that it will be a long time before one can be developed. In the meantime, the step-by-step progress made in recent years should be continued. Indeed, any fixed policy may be undesirable because of the constantly changing conditions in the United States.

CLARENCE S. JARVIS,[31] M. ASCE.—Relatively few water storage reservoir projects of large capacity and major importance have been developed to serve a single purpose. Ordinarily one principal interest was stressed, with incidental benefits to others, and possibly some disadvantages and inconvenience accrued unless prior rights were recognized and fully protected.

The legislation embodied in *House Document No. 308*,[32] which was enacted into law in Section 1 of the River and Harbor Act of January 21, 1927, initiated a new era of practical hydrology. It directed the Chief of Engineers to prepare technical reports on the principal river systems of the United States, except the Colorado River (on which the Bureau of Reclamation, United States Department of the Interior, had already made notable progress), from the fourfold standpoints of navigation, flood control, power development, and irrigation. Naturally, the interests of municipal water supply, manufacturing industries, recreation, fish and game life, and other outstanding items of national life and economic structure were duly investigated and included in the final reports. The impressive array of "House Document No. 308 Reports," published during the following decade, helped to establish the concept of "multiple-purpose reservoirs," in which both the principal and the incidental beneficiaries were recognized, along with their eligibility and probable ability to share in the capital cost of the undertakings. This new concept has made possible the launching of several important key projects in the realms of flood control, navigation, hydroelectric power development, irrigation, and municipal water supply. Without the hydroelectric energy made available by such projects as Grand Coulee Dam (on the Columbia River), Bonneville Dam (on the Columbia River), Hoover Dam (on the Colorado River), the notable features of the Tennessee Valley Authority, and other monumental achievements, just how could this nation have accomplished its tasks during the years of World War II?

If a substantial part of the total costs of such projects (with their demonstrated strategic value and importance during national crises) must be charged to the general welfare, while subscribers under irrigation, municipal water supply, and hydroelectric power projects pay all that can rightfully be assessed against them, such a basis of financing is both justifiable and commendable.

Alleged conflicts of major interests in a multiple-purpose reservoir project are too often grossly overdrawn. Conceding that the flood control ideal inclines toward an empty reservoir at certain seasons, and that hydroelectric power interests prefer a high storage head, both ideals must readily submit to limitations and to modifications in their own behalf, and should make even greater concessions in the spirit of compromise to promote conservation and utilization of basic natural resources. If properly designed and operated, a reservoir filled nearly to spillway level is proof of a mission accomplished for that particular season or period. It is still capable of storing and automatically releasing a regulated volume of floodwater over the spillway, with but minor or negligible damage to the pasture or woodland below the designed high-water contour, for which flowage rights are already secured. A remarkably large

[31] Cons. Engr., Salt Lake City, Utah.

[32] *House Document No. 308* 69th Cong., 1st Session, 1928.

proportion of the total reservoir capacity is ordinarily located above the spillway level. With assurance that automatic release will continue at the predetermined rates, over the open spillway, the operators are relieved of needless worries and the pitfalls of immature decisions regarding emergency drawdown.

Recent advances in the field of hydrometeorological services by the United States Weather Bureau and cooperating agencies have effectively removed the probability, if not the possibility, of unwarned approach of catastrophic storms (and resulting floods) except those associated with structural failures. This situation certainly justifies the adoption of an intelligent and liberal policy in the field of reservoir drawdown. With long-range weather forecasts attaining higher percentages of accuracy and dependability, it is becoming more justifiable to maintain considerable storage volume of floodwater for orderly release to promote beneficial use. On the other hand, demand for hydroelectric energy may justify a drawdown below the ideal storage level when there is reasonable assurance of increased volume of river discharge in the near future, capable of replenishing the storage to the desired level or even above that level.

Some of the most important and valuable storage capacity of a reservoir may consist of what is known as bank storage. Lateral infiltration into coarse granular deposits or other readily permeable geological formations above Elephant Butte Dam on the Rio Grande in New Mexico accounted for a return flow, during the final drought year of a series ending in 1940, which was sufficient to offset all reservoir losses and to yield an additional dividend of approximately 47,800 acre-ft. The 46-year mean annual loss had been calculated as 198,700 acre-ft from records of the Bureau of Reclamation, as summarized, in tabular form[33] and graphical form,[34] by the writer in 1943. In some cases, geological reconnaissance may determine the advisability of opening trenches, cuts, or shafts to penetrate the potential underground storage strata, especially where the topsoil mantle would retard the inflow and outflow unduly.

The Symposium is of comprehensive scope and constitutes a valuable contribution to the subject of multiple-purpose reservoirs.

[33] "Supplementary Gleanings from the Field of Hydrology," by Clarence S. Jarvis, Brentano's Inc., Washington, D. C., 1943, pp. 40–41.
[34] *Ibid.*, p. 37.

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DISCUSSIONS

AN ENGINEERING CONCEPT OF FLOW IN PIPES

Discussion

BY ANDRE L. JORISSEN, WESTON GAVETT, AND ARTHUR L. COLLINS

ANDRE L. JORISSEN.[44]—A worthwhile endeavor to approach the problem of flow in pipes of natural roughness synthetically is made by Mr. Harris. The main obstacle to a comprehensive solution lies in the complexity of the roughness pattern and in the resulting difficulty of defining, and hence of measuring, roughness.

An attempt is made to characterize roughness by a specific roughness coefficient. It should be understood that this coefficient is the constant friction factor of fully turbulent flow in a pipe of unit diameter, for a given roughness pattern. No specification of the unit length used is necessary if it is stated that the value of D in Eq. 3 must be expressed in the same units.

The analysis of transition from laminar to fully turbulent flow is based on Eq. 1, derived from the author's experimental study of a smooth brass pipe. This formula (Eq. 1) is similar to a relation which was propounded, as early as 1922, by Charles Hanocq[45] of Liege, Belgium:

$$f = 0.00668 + \frac{0.532}{\sqrt[3]{R}} \qquad (24)$$

The Hanocq formula has been widely and satisfactorily used in western Europe since the early 1920's. It is based on numerous experiments, both with air and water and carried up to the Reynolds numbers of 500,000. Also, it has the advantage of a pure cube root in the computation of the second term.

A Reynolds number of 3,240,000 having been reached in the Nikuradse experiments, it is desirable to investigate to what extent classical formulas agree with experimental results. Typical formulas for smooth pipes (in addition to the Hanocq formula already mentioned) are as follows:

NOTE.—This paper by Charles W. Harris was published in May, 1949, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1949, by E. L. Harrington; October, 1949, by Ralph W. Powell, J. C. Stevens, George A. Whetstone, G. H. Hickox, and James F. Halsey and William M. Owen; and November, 1949, by Tsen-Ding Chen, and Lynn Perry.

[44] Associate Prof. of Civ. Eng., Pennsylvania State College, State College, Pa.

[45] "A propos du calcul des pertes de charge dans les conduites forcées," by Charles Hanocq, *Revue Universelle des Mines*, Liège, Belgium, February, 1922, Series 6, pp. 217–234.

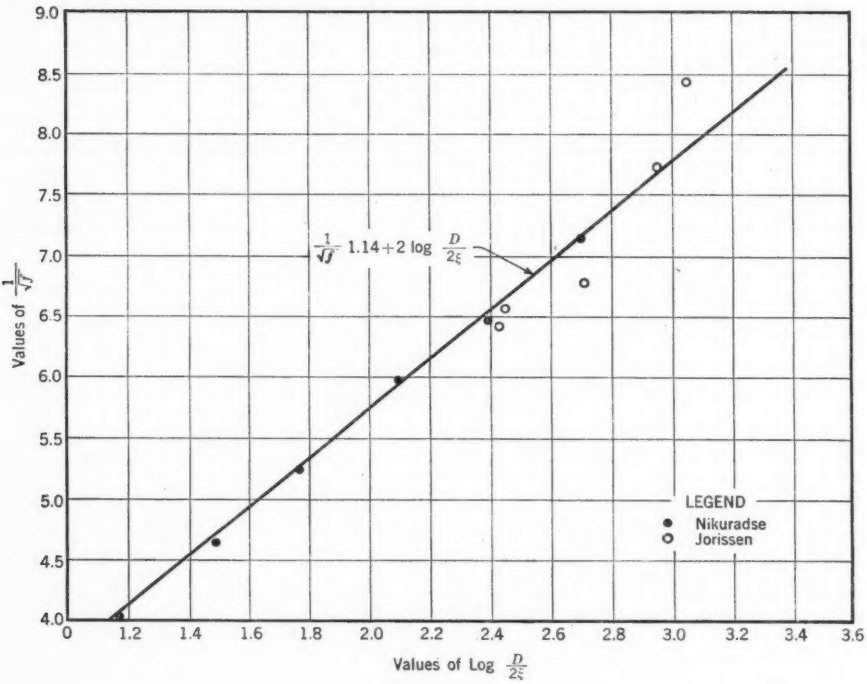


FIG. 22

The Blasius[46] formula (1913)—

$$f = \frac{0.3164}{R^{0.25}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

for $R \leqq 100{,}000$; the Lees formula[12] (1915), based on experiments by T. E. Stanton and J. R. Pannell[13]—

$$= 0.0072 + \frac{0.6104}{R^{0.35}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

for $R \leqq 460{,}000$; and the Schiller-Hermann[47] formula (1930)—

$$f = 0.0054 + \frac{0.396}{R^{0.3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

[46] "Das Achnlichkeitsgesatz bei Reibungsvorgängen in Flüssigkeiten," by H. Blasius, *Forschung auf dem Gebiete der Ingenieurwesens*, No. 131, 1913.

[12] "On the Flow of Viscous Fluids Through Smooth Circular Pipes," by Charles H. Lees, *Proceedings*, Royal Soc. of London, Series A, Vol. 91, 1915, p. 46.

[13] "Similarity of Motion in Relation to the Surface Friction of Fluids," by T. E. Stanton and J. R. Pannell, *Philosophical Transactions*, Royal Soc. of London, Series A, Vol. 214, 1914, p. 199.

[47] "Widerstand von Platte und Rohr bei hohen Reynoldsschen Zahlen," by L. Schiller and R. Hermann, *Ingenieur Archiv*, Vol. I, 1930, p. 392.

—which gives results in good agreement with the Nikuradse experiments up to a Reynolds number of 2,500,000. A discrepancy is however apparent for $R < 50{,}000$. The Nikuradse[48] formula (1932)—

$$f = 0.0032 + \frac{0.221}{R^{0.237}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

is valid for a Reynolds number greater than 100,000, which is the limit of validity of the Blasius formula.

TABLE 8.—PERCENTAGE COMPARISON OF FORMULAS FOR FRICTION LOSS

Reynolds' No. (R) (thousands)	f by Eq. 29 (writer)	BLASIUS		HANOCQ		NIKURADSE		SCHILLER-HERMANN		HARRIS	
		f_B, Eq. 25	$\frac{f-f_B}{f}$	f_a, Eq. 24	$\frac{f-f_a}{f}$	f_N, Eq. 28	$\frac{f-f_N}{f}$	f_S, Eq. 27	$\frac{f-f_S}{f}$	f_H, Eq. 1	$\frac{f-f_H}{f}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
2	0.0490	0.0473	+5.51	0.0491	−0.20	0.0397	....	0.0459	+6.34	0.04970	−1.43
3	0.0437	0.0428	+2.06	0.0437	0	0.0363	....	0.0413	+5.50	0.04420	−1.14
5	0.0380	0.0376	+1.05	0.0379	+0.26	0.0326	....	0.0362	+4.75	0.03830	−0.79
10	0.0315	0.0316	−0.32	0.0315	0	0.0281	....	0.0304	+3.50	0.03165	−0.48
15	0.0284	0.0286	−0.71	0.0283	+0.35	0.0258	....	0.0275	+3.17	0.02840	0
20	0.0264	0.0266	−0.76	0.0263	+0.38	0.0243	....	0.0257	+2.65	0.02635	+0.19
25	0.0250	0.0252	−0.80	0.0249	+0.40	0.0233	....	0.0244	+2.40	0.02490	+0.40
50	0.0211	0.0212	−0.47	0.0212	−0.47	0.0202	....	0.0208	+1.42	0.02101	+0.43
100	0.0180	0.0178	+1.11	0.0182	−1.11	0.0176	+2.22	0.0179	+0.56	0.01795	+0.28
150	0.0165	0.0161	+2.42	0.0167	−1.21	0.0163	+1.21	0.0165	0	0.01643	+0.42
200	0.01553	0.0150	+3.42	0.01582	−1.86	0.01545	+0.51	0.01557	−0.26	0.01553	0
250	0.01485	0.0142	+4.38	0.01513	−1.89	0.01482	+0.20	0.01491	−0.40	0.01484	+0.07
300	0.01432	....	....	0.01466	−2.37	0.01432	0	0.01441	−0.63	0.01433	−0.07
350	0.01390	....	....	0.01426	−2.59	0.01393	−0.22	0.01400	−0.72	0.01392	−0.14
400	0.01355	....	....	0.01393	−2.81	0.01355	−0.30	0.01366	−0.81	0.01356	−0.07
500	0.01300	....	....	0.01341	−3.16	0.01306	−0.46	0.01313	−1.00	0.01304	0
1,000	0.01151	....	....	....	....	0.01156	−0.43	0.01168	−1.48	0.01160	−0.78
1,500	0.01079	....	....	....	....	0.01080	−0.09	0.01096	−1.58	0.01090	−1.02
2,000	0.01032	....	....	....	....	0.01030	+0.19	0.01050	−1.74	0.01046	−1.36
2,500	0.00999	....	....	....	....	0.00993	+0.60	0.01017	−1.80	0.01015	−1.61
3,000	0.00974	....	....	....	....	0.00965	+0.92	0.00991	−1.75	0.00991	−1.75
3,500	0.00954	....	....	....	....	0.00941	+1.37	0.00971	−1.78	0.00972	−1.89
4,000	0.00937	....	....	....	....	0.00922	+1.60	0.00954	−1.82	0.00957	−2.14
5,000	0.00910	....	....	....	....	0.00891	+2.08	0.00927	−1.87	0.00932	−2.42

More recently, the writer has derived a comprehensive formula based on the Nikuradse measurements of velocity distribution and pressure loss. This formula[49] is expressed as

$$f = 0.0055 + \frac{0.4886}{R^{0.3183}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

and checks remarkably well with the experiments of Nikuradse and the writer for all values of the Reynolds number.

A comparison of the various formulas (see Table 8) shows fairly good agreement in the range of validity of each formula, but the agreement is not extended over this range. Eqs. 1 and 29 give close values of f up to the high

[48] "Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren," by J. Nikuradse, *Forschungsheft*, Verein Deutscher Ingenieure, No. 356, September–October, 1932.

[49] "Contribution à l'étude des pertes de charge continues dans les conduites circulaires," by Andre L. Jorissen, *Bulletin du Centre d'Etudes, de Recherches et d'Essais Scientifiques des Constructions du Génie Civil et d'Hydraulique Fluviale*, Liège, Belgium, 1949.

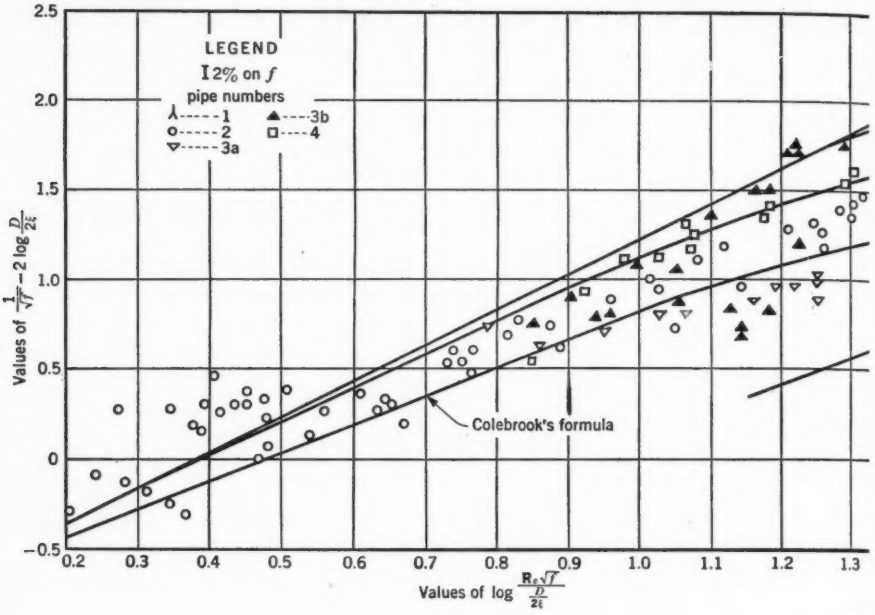
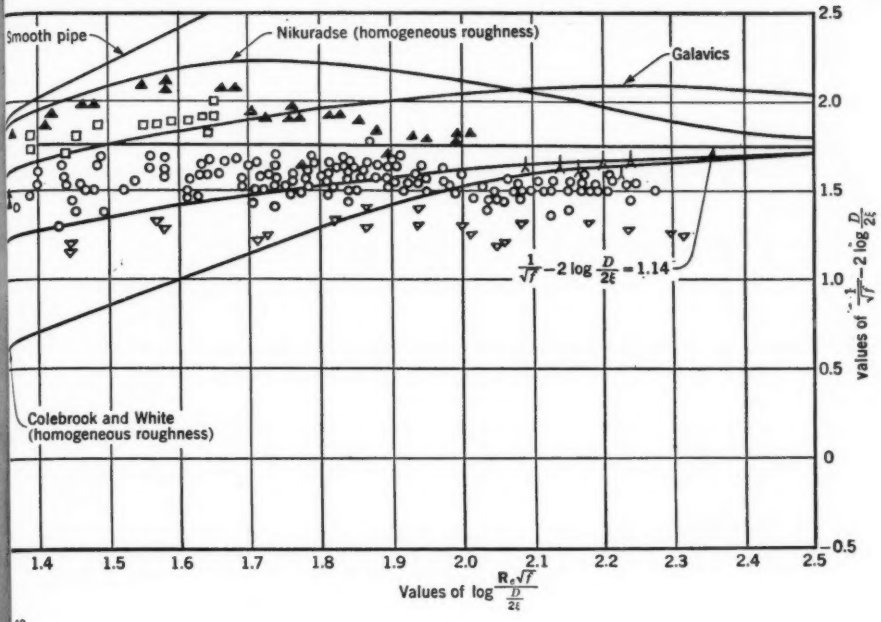


FIG.

Reynolds numbers, a slight discrepancy being noticeable in the extreme values of the latter parameter. However, for all practical purposes, the two formulas may be said to agree.

The proposed method of characterizing roughness is based on the consideration of the specific roughness coefficient. It is doubtful, however, if such a coefficient will prove satisfactory in the study of commercial rough pipes. In the experience of the writer, the mean height of roughness protuberances may vary considerably from one pipe to another, even for new pipes of approximately equal diameters manufactured in the same factory. This opinion was obtained by measuring the mean roughness height ϵ by a pneumatic feeler.[49] It was shown that, in the new industrial steel pipes studied, ϵ is a characteristic value for a given pipe. The value of ϵ may vary from one cross section to another and from point to point in one section but, by averaging a large number of measurements, a characteristic mean value is obtained. As stated before, ϵ differs considerably from one pipe to another. This stresses the difficulty of using any of the classical formulas for rough pipes and seems to restrict the use of the specific roughness coefficient. Pneumatic feeler measurements, on the contrary, are easier to make than pressure loss determinations and may further be repeated from time to time to follow the evolution of the pipe roughness.

Another conclusion of the aforementioned study was that, in the zone of fully developed turbulence, the value of f was given with a very satisfactory



23

approximation by the Nikuradse formula:

$$\frac{1}{\sqrt{f}} = 1.14 + 2 \log \frac{D}{\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

in which ϵ was the mean roughness height, as measured by the pneumatic feeler (Fig. 22).

A smooth transition in the law of variation of f as a function of the Reynolds number was recorded. This is also in agreement with the Colebrook and White experiments. In this zone, the law of variation for industrial pipes is situated between the Nikuradse law for homogeneous roughness and the Colebrook and White law for heterogeneous roughness. The writer's experiments show a classification following the roughness homogeneity (Fig. 23).

WESTON GAVETT,[50] M. ASCE.—The book on "Hydraulic Tables"[8] by Gardner S. Williams and Allen Hazen, Members, ASCE, has been described by the author (under the heading, "Inadequacy of Some Early Methods") as "based on a fallacy that should have been apparent at the time." Mr. Harris further remarks that:

> "A constant-exponent formula was developed, selecting only the data from the smoothest pipes attainable, the exponent of V reducing to 1.85. It was then assumed that all rough pipes would follow that same exponential 'law' * * *."

[50] Associate, Clyde Potts, New York, N. Y.

[8] "Hydraulic Tables," by Gardner S. Williams and Allen Hazen, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1920.

In the Introduction[51] to "Hydraulic Tables" Messrs. Williams and Hazen pointed out that—

> "* * * the values of the exponents vary with different surfaces and also their values may not be exactly the same for large diameters and for small ones, nor for steep slopes and for flat ones. * * *
>
> "The exponents in the formula used were selected as representing as nearly as possible average conditions, as deduced from the best available records of experiments upon the flow of water in such pipes and channels as most frequently occur in water-works practice."

That the Hazen-Williams formula in fact represents a fair average for pipes in water systems is indicated by its continued use and the results of many tests before and after its publication. The convenience of such an exponential formula for solving network problems, using the "Hydraulic Tables" and the Hazen-Williams slide rule is generally recognized.

The exponent of V of 1.85, between the usual values of 1.75 for smooth pipe and 2.00 for complete turbulence, seems to represent a most probable value. With the increasing use of smoother pipe surfaces, the exponent 1.85 may become high rather than low.

For important transmission mains, particularly those of large diameter, the water works engineer will study all available data and theory; but, for distribution system studies, the Hazen-Williams formula should continue as a favorite tool.

ARTHUR L. COLLINS,[52] ASSOC. M. ASCE.—The writer's concept of the flow of water in pipes is best illustrated by a multitude of points on a logarithmic chart. They are so numerous that they outline a band. Each point has been established by two readings, velocity and loss of head, as reported by an observer.

Each new formula proposed for pipe flow as the result of a laboratory test is definitely fixed by a line within the chart band. There can be about as many different formula exponents as there are stars in the Milky Way. Regardless of the formula used the data are someplace on the chart and a choice can be made after reading the testimonials.

Other factors enter into flow measurements and characteristics in the field which are not present in the laboratory where the setup includes a small pipe with perfect bore. Undulations and ovalness (out of round) are conditions that can have a marked effect upon energy losses.

The study of the velocity distribution or velocity factor should be included in pipe experiments as a method of classifying pipe conditions and making the data more comparable. It is reasonable to assume that at some time the pipe line will be designed to provide a certain velocity factor which will indirectly insure maximum transportation efficiency.

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[51] "Hydraulic Tables," by Gardner S. Williams and Allen Hazen, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1920, p. 1.

[52] Cons. Engr., Berkeley, Calif.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MINERAL LAND SURVEYS

Discussion

BY ARTHUR J. MCNAIR

ARTHUR J. MCNAIR,[31] Assoc. M. ASCE.—In this paper the writer presented material on the history and methods of making mineral land surveys because he felt that the subject had not received any attention from the engineering profession since the turn of the twentieth century. The comments emphasize the timeliness of the subject because the old-timers familiar with the procedures peculiar to this type of surveying are rapidly diminishing in numbers. Furthermore, the number of both patented and unpatented mining claims has been increasing continually during these years while at the same time the corner monuments fixing their positions on the ground have become increasingly difficult to find.

This situation, as stated by Mr. Meldrum, creates a fertile field for the development of young engineers. As large sections of the old mining districts are consolidated for unified operation by larger companies, the need for resurvey of the old claims, together with the location and patenting of new claims to cover the vacant fractions, is immediately felt. It is of more than passing interest to note that titles to claims in an old established mining district are usually in such a condition that at least two to three years of work is required by legal counsel before a mining company can consolidate ownerships and begin operations.

The discussions indicate that there are continual changes in the administration of the lands in the public domain and that these have produced some changes in the procedure of making mineral land surveys. A substantial contribution to this paper has been made by the discussions which describe the functions of the Bureau of Land Management, United States Department of the Interior, which now administers the lands of the public domain.

Because the discovery of gold—and other mineral values—was frequently the reason for the early settlement of many of the western states, and because each mining district grew up autonomously, it is not surprising that today

NOTE.—This paper by Arthur J. McNair was published in November, 1948, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1949, by Charles P. Eager, and Robert Lenon; and June, 1949, by John V. Meldrum, and Arthur D. Kidder.

[31] Associate Prof. of Civ. Eng. and Head of Surveying Dept., Cornell Univ., Ithaca, N. Y.

there are local variations in mining laws and regulations. The hope of the writer that these variations would be presented by those discussing the paper, thereby making it more applicable to the entire United States, has been at least partly fulfilled. Since California probably does more placer mining than any other state, the discussions presenting the variations in placer surveys in that state as compared to others are important. It is interesting to note that the possessory right to unpatented claims in the form of locations is taxable in California as this sets up different economic risks for a locator than are established where locations are not taxed.

The requirement in Arizona that "* * * end-center monuments (or their counterparts at each end of the lode line when the lode does not cut the end line exactly at its center) must be set * * *" differs from the procedure in other states. The writer cannot understand the necessity for end centers. Certainly they serve no useful purpose, and one cannot help but surmise what the mineral surveyor might designate as the lode line for a claim located in a region of disseminated copper porphyries where there are no precise boundaries, such as in Arizona, or in a region of sedimentary or so-called blanket deposits such as occur in so many places in the western states.

With the introduction of European-make transits into the United States, a considerable part of the writer's discussion concerning the precision of mineral land surveys will be modified. Some of these instruments are not only much lighter and easier to carry but are more accurate and much easier to read than the conventional American transits described in the paper. In no phase of surveying are portability, accuracy, and ease of reading more essential than in mineral surveying, and the next few years will doubtless see an improvement in procedures corresponding to the improvement in instruments.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DIFFUSION OF SUBMERGED JETS

Discussion

BY HAROLD R. HENRY

HAROLD R. HENRY,[31] JUN. ASCE.—The value of this paper lies not only in the specific results obtained, but in the method of attack, which may be applied with modification if necessary to submerged jets of a different nature and application. For instance, one of the specific examples referred to by the authors, efflux from a submerged sluice gate, is cited for comparison. The upper limit of submergence of a sluice gate (that is, infinite vertical extent of tailwater) may be represented by substituting a boundary for the plane of symmetry of the completely submerged two-dimensional jet shown in Figs.

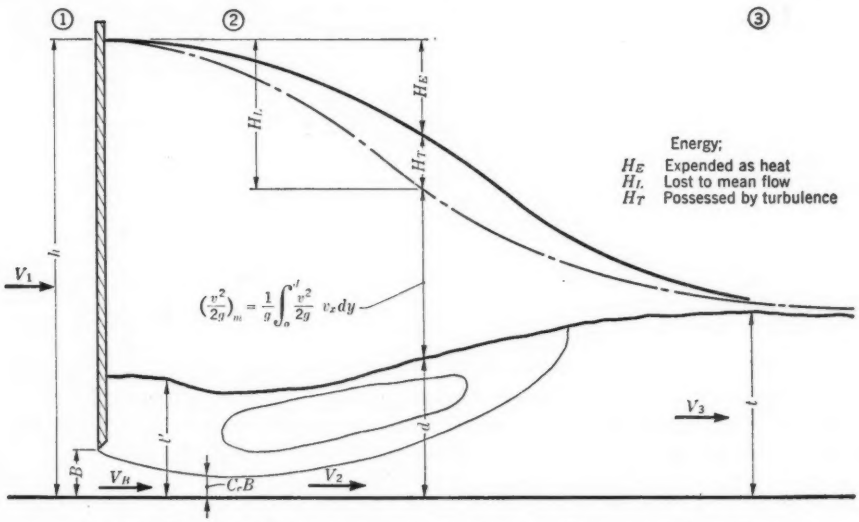

FIG. 31.—SCHEMATIC REPRESENTATION OF FLOW FROM A SUBMERGED SLUICE GATE

NOTE.—This paper by M. L. Albertson, Y. B. Dai, R. A. Jensen, and Hunter Rouse was published in December, 1948, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1949, by James S. Holdhusen, Duilio Citrini, and Stanley Corrsin; and September, 1949, by W. Douglas Baines, and Abraham Streiff.

[31] Instr., Civ. Eng. Dept., Georgia Inst. of Technology, Atlanta, Ga.

1 and 21, whereas the lower limit is represented by a free hydraulic jump forming on this boundary downstream from the efflux section. By far the majority of cases involving flow under a sluice gate occur nearer the limiting case of the free jump: Fig. 31 represents, schematically, a typical intermediate condition.

The authors' investigation serves to indicate the flow pattern for conditions approaching the upper limit. On the other hand, at lower degrees of submergence, the proximity of the free surface cannot be ignored. The new effects introduced are: (*a*) A definite gravitational effect as is evidenced by the depression of the surface indicated in Fig. 31, and which may be considered in the analysis by the inclusion of a Froude number; and (*b*) the limiting of the fluid downstream from the efflux section to a finite vertical extent which calls

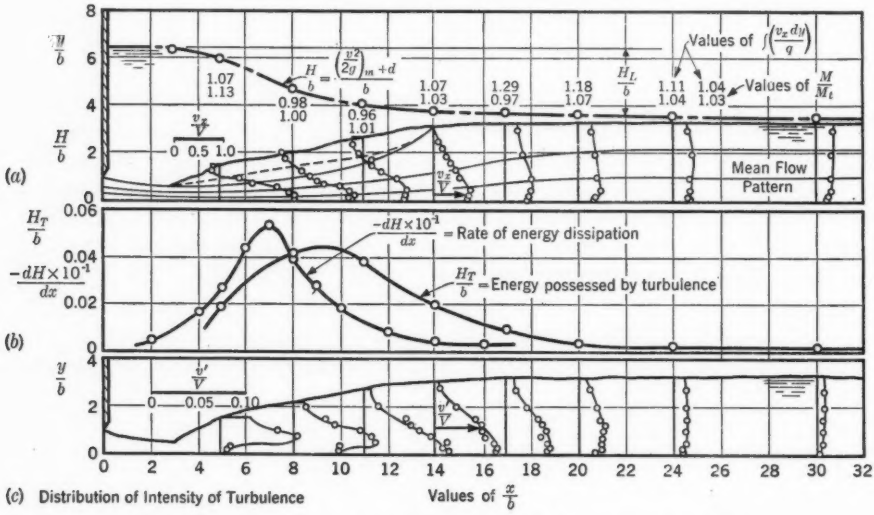

FIG. 32.—PATTERNS FOR FREE JUMP: $F_B = 2.0$; $F_1 = 4.30$; $t/B = 3.35$; $h/B = 6.35$; $C_c = 0.60$; AND $C_d = 0.561$

for a variable describing the relative depth of the tailwater. The foregoing variables may be $F_B = \frac{q}{B\sqrt{g\,B}} \rightarrow$ and $\frac{t}{B}$, in which B is the height of gate opening ($=B_0/2$); t is the tailwater depth; q is the mean flow per unit width or gate opening; and g is the acceleration of gravity. Hence, when modified to apply to two-dimensional flow from a submerged sluice gate, the formula for relative velocity at any point in the flow (Eq. 1) would take the form:

$$\frac{v}{V_B} = f\left(F_B, \frac{t}{B}, \frac{x}{B}, \frac{y}{B}\right) \ldots\ldots\ldots\ldots\ldots\ldots (84)$$

in which $V_B = q/B$, the mean velocity at the gate.

Eq. 84 and the reasoning leading to this equation indicate that an orderly method to determine experimentally the effect of proximity of the free surface

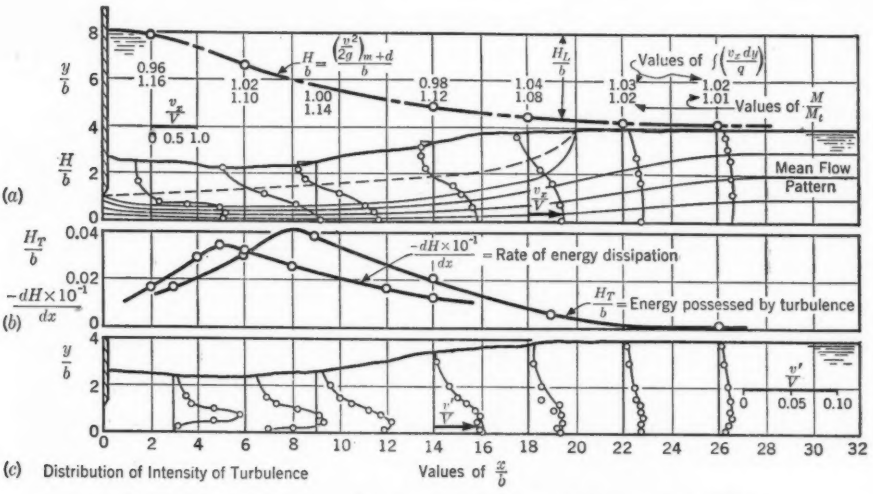

FIG. 33.—PATTERNS FOR $t/B = 4$; $F_B \doteq 2.0$; $h/B = 7.95$; AND $C_d = 0.501$

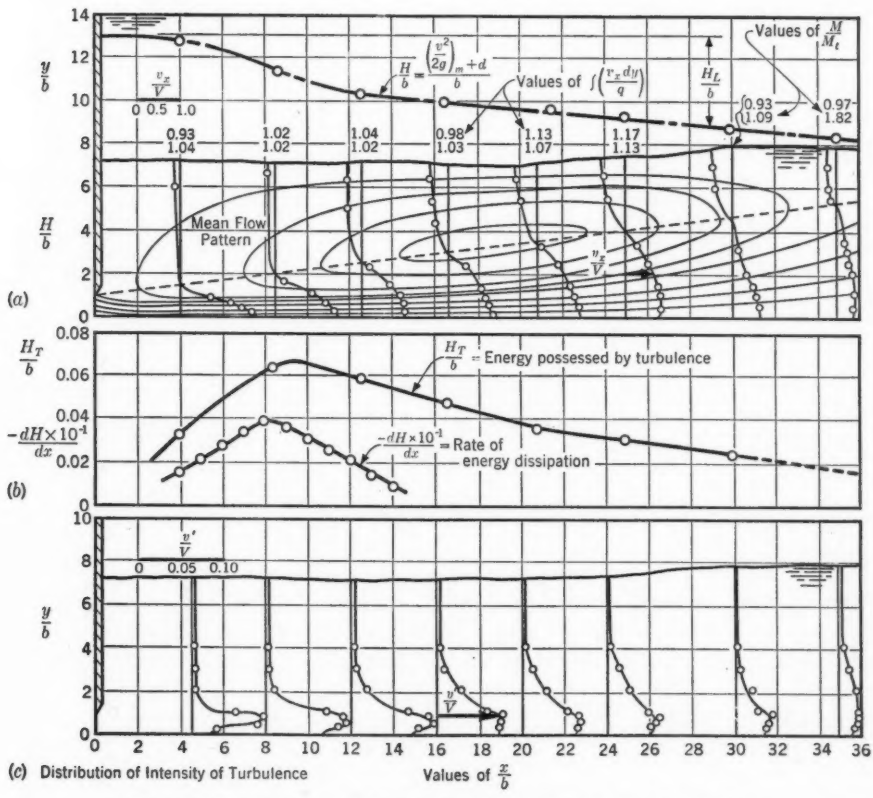

FIG. 34.—PATTERNS FOR $t/B = 8$; $F_B = 2.0$; $h/B = 13$; AND $C_d = 0.391$

would be to investigate the flow phenomena with systematic variation of F_B and t/B. The writer has conducted a series of tests to examine the mean flow pattern and the intensity of turbulence (v', the root-mean-square of the velocity fluctuation) for the three conditions holding $F_B = 2$—namely, $t/B = 3.35$, 4, and 8. The results are presented in Figs. 32, 33, and 34. The first value of t/B was necessary for the formation of a free jump at the vena contracta; the other values were chosen to indicate the effects of increased submergence.

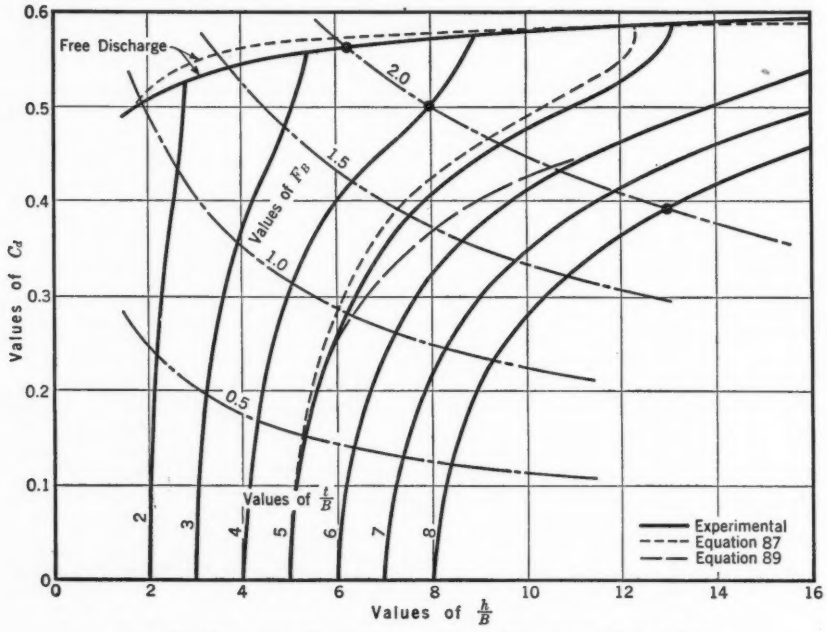

FIG. 35.—DISCHARGE CHARACTERISTICS OF A SUBMERGED SLUICE GATE

It is enlightening to consider the effect of submergence from the standpoint of the discharge characteristics of the gate. Simultaneous consideration of F_B and the headwater depth, h, leads to the expression:

$$F_B = \left(\sqrt{\frac{2h}{B}}\right)\frac{q}{\sqrt{2gh}} = \sqrt{\frac{2h}{B}}\,C_d \quad \ldots\ldots\ldots\ldots (85)$$

in which $C_d = q/\sqrt{2gh}$ may be treated as a discharge coefficient. Thus, whenever F_B appears in any functional expression, as Eq. 84, it may be replaced by C_d or h/B. Hence, the variation of C_d and h/B at various degrees of submergence should be noted. A dimensionless expression for the experimental determination of this variation is

$$C_d = f\left(\frac{h}{B}, \frac{t}{B}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (86)$$

The discharge characteristics of a sharp-edged sluice gate, as determined

experimentally by the writer, are shown as the solid curves in Fig. 35. The center-line curves, lines of constant F_B, were computed according to Eq. 85. The three solid points represent values at which the patterns of mean flow and turbulence were investigated, and serve to indicate clearly the change in degree of submergence.

The experimental trend of the curves, C_d versus h/B, in Fig. 35 was verified by an approximate theoretical approach, utilizing the authors' assumption of hydrostatic pressure distribution throughout the flow plus the assumptions (see Fig. 31): No energy loss between sections 1 and 2; uniform velocity distribution at sections 1 and 3; and, at the vena contracta (section 2), zero velocity in the roller and uniform velocity in the jet. The energy relation between sections 1 and 2 and the momentum relation between sections 2 and 3 allow the following equations to be written:

$$C_d = K\,C_c\sqrt{1-\frac{t'}{h}} \qquad (87)$$

and

$$\frac{t'}{b} = \frac{K^2 \pm \sqrt{K^4 + 4\,A\left[A\left(\frac{t}{B}\right)^2 - \left(\frac{h}{B}\right)K^2\right]}}{2\,A} \qquad (88)$$

in which $K = \dfrac{1}{\sqrt{1-\left(\dfrac{C_c B}{h}\right)^2}}$ and $A = \dfrac{1}{4\,C_c\left(1-\dfrac{C_c B}{t}\right)}$. Values of C_d given by the resulting forms of Eq. 87 for $t/B = 5$ and for free discharge (assuming $C_c = 0.6$) are plotted in Fig. 35 as the dotted curves. The dashed curve represents the results of applying, for $t/B = 5$, the common formula:

$$C_d = C_c\sqrt{1-\frac{t}{h}} \qquad (89)$$

which is really the limiting form of Eq. 87 as the depression in the water surface decreases and t' approaches t.

The writer's data were obtained in the laboratory of the Iowa Institute of Hydraulic Research (State University of Iowa, Iowa City), by using a sharp-edged sluice gate in a 1-ft flume which received water from a constant-head tank. Velocities were determined by pitot tubes of the Prandtl and cylindrical type, the latter being used in regions of inclined flow. The fluctuations in velocity due to turbulence were determined by a turbulence pitot[26,32] (capable of indicating only the logitudinal component), especially constructed at the Iowa Institute for such measurements in water. The results of investigating the internal flow phenomena are shown in Figs. 32, 33, and 34. The basic data of these diagrams are the surface profiles, the velocity distributions, and

[26] "The Development of a Turbulence Pitot for Use in Water," by D. W. Appel, thesis presented to the State University of Iowa, at Iowa City, Iowa, in June, 1949, in partial fulfilment of the requirements for the degree of Master of Science.

[32] "An Evaluation of Several Experimental Methods of Measuring Turbulence in Flowing Water," by P. G. Hubbard and J. S. Norton, *Report No. 2*, rept. prepared by the Iowa Inst. of Hydr. Research for the Office of Naval Research under contract N8onr-500, February, 1949.

the distributions of the intensity of turbulence. All other data are computed from them. As compared to Figs. 1 and 21, Figs. 32, 33, and 34 indicate vividly the effect of the proximity of the free surface.

As a direct result of limiting the vertical extent of the fluid, three basic changes occur in the flow phenomena:

(1) The discharge or volume flux past successive sections perpendicular to the jet, instead of increasing continually according to Eqs. 42 and 44, must be constant and equal to the discharge at the gate—that is,

$$\frac{\int_0^d v_x \, dy}{q} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (90)$$

in which d is the depth at any section;

(2) As a direct consequence of item (1), fluid undergoing the induced motion because of the accelerative action of the jet must be replenished by a return flow both at and near the surface, forming what is termed the "roller"; and

(3) The vertical expansion of the jet does not continue indefinitely but is limited by the intersection of the uppermost streamline of the jet with the free surface. Inasmuch as the free surface is a line of constant pressure, this intersection must be at right angles, necessitating a curvilinear pattern of jet diffusion instead of the linear pattern indicated in Figs. 1 and 21. The "nominal boundary of the jet," as indicated by the authors, corresponds closely to the lines of zero horizontal velocity in Figs. 33, 34, and 35.

The depression and unevenness of the surface, due to the gravitational effect, make invalid the reasoning leading to Eq. 6. Rather they lead to the constancy of the sum of pressure forces on any vertical section and the flux of the longitudinal component of momentum at the same section. Again, using the authors' assumption of hydrostatic pressure distribution (which is the only one of the three original assumptions that could reasonably apply to flow conditions with finite tailwater depth) and letting M represent the indicated sum, the following relation may be written:

$$\frac{M}{M_t} = \frac{\frac{1}{2}\gamma d^2 + \rho \int_0^d v_x^2 \, dy}{\frac{1}{2}\gamma t^2 + \rho q V_t} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (91)$$

in which γ is the specific weight of water. The subscript t in Eq. 91 applies to uniform tailwater conditions.

Actual values of the ratios presented in Eqs. 90 and 91 have been computed from the experimental data and appear above the mean flow patterns. The constant value of unity is indicated; the variations are no doubt due in part to air entrainment and to accelerative forces within the roller.

With the complex flow pattern exhibited downstream from a submerged sluice gate, one could not expect relations in the nature of Eqs. 43 and 45 to represent energy flux at any section. Instead, the writer has shown variation

in total head, H, plotted above the mean flow patterns, computed from the experimental data as follows:

$$H = d + \frac{1}{q}\int_0^d \frac{v^2}{2g} v_x \, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots (92)$$

Here the authors' assumption of hydrostatic pressure distribution is again utilized.

At any section the energy lost to the mean flow, as referred to the energy at the efflux section, is the sum of the kinetic energy of turbulence and the energy expended as heat over the preceding distance. This is the relation indicated by the authors (under the heading, "Statement of the Problem")—

> "* * * the kinetic energy of the oncoming flow steadily [is] * * * converted into kinetic energy of turbulence, and the latter steadily * * * [decays] through viscous shear."

This may be expressed in the equation:

$$H_L = H_E + H_T \quad \ldots\ldots\ldots\ldots\ldots\ldots (93)$$

in which H_L is the energy lost to the mean flow; H_T is the kinetic energy of turbulence; and H_e is the energy already expended as heat. This relation is represented schematically in Fig. 31.

The actual values of H_T were computed from the experimental data by the following equation:

$$H_T = 3\frac{1}{q}\int_0^d \frac{(v'_x)^2}{2g} v_x \, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots (94)$$

In Eq. 94 the assumption of equal turbulence components in the three coordinate directions has been made. The values of H_T so computed were too small to allow satisfactory representation as indicated in Fig. 31, but necessitated an expanded scale as plotted in Figs. 32, 33, and 34. The discovery that the energy of the turbulence, which is involved in so great a dissipation of the mean flow energy, is on the order of only 1% as great as the total energy lost was one of the most interesting results of the writer's experiments.

To allow rapid and vivid comparison, $-\frac{dH}{dx}$ (the rate at which energy is lost to the mean flow) was obtained graphically as the slope of the total head line and plotted on the same set of coordinates as H_T. Because of the smallness of H_T this rate represents essentially the rate that energy is converted into heat. If the scale of the smallest turbulent eddies were constant throughout the flow, $-\frac{dH}{dx}$ and H_T should attain maximum values at identical sections.[27] Since the writer had no way of determining the scale, such a statement cannot be made. The problem of energy dissipation due to turbulence will not be completely solved until the discovery of some method of evaluating the size of these smallest eddies.

[27] "Statistical Theory of Turbulence," by G. I. Taylor, *Proceedings*, Royal Soc. of London, Series A., Vol. 151, 1935, pp. 421–478.

The writer has concluded that the method of experimental attack as applied in the paper is of great value in investigating any type of submerged jet. It only behooves the investigator to recognize such factors as the dissimilarities resulting from boundary conditions, and to modify the approach accordingly.

Acknowledgment.—This discussion is based on a thesis by the writer[33] and part of the data for the mean flow patterns for $F_B = 2.00$, $t/B = 4$, and $t/B = 8$ were obtained from a thesis by H. K. Liu.[34]

[33] "A Study of Flow from a Submerged Sluice Gate," by Harold R. Henry, thesis presented to the State University of Iowa, at Iowa City, Iowa, in February, 1950, in partial fulfilment of the requirements for the degree of Master of Science.

[34] "Diffusion of Flow from a Submerged Sluice Gate," by H. K. Liu, thesis presented to the State University of Iowa, at Iowa City, Iowa, in February, 1949, in partial fulfilment of the requirements for the degree of Master of Science.

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